Véletlen Cantor-halmazok különbsége: Az általános eset

Algebraic difference of random Cantor set: The general setup

Technical report

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Preface

This technical report consists of a paper that the authors Michel F. Dekking[1], Karoly Simon[2] and Balázs Székely will submit to one of the top journal of the field of probability theory in the short future.

The paper provides conditions for having interval almost surely in the difference of two random Cantor sets, where the Cantor sets come from a very broad family. Basically, the requirement is that the similarity dimension of the random Cantor set has to be larger than one half. The result presented here is the most general that can be achieved for Cantor sets with non intersecting cylinders.

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Algebraic difference of random Cantor set

1 Introduction and main results

In 1987, Palis and Takens [9] studying the dynamical behavior of diffeomorfism presented a conjecture about the size of the algebraic difference of two Cantor sets. Informally, if the size of the difference is large then it contains an interval. More precisely, if C_1 and C_2 are two Cantor sets then the algebraic difference

$$C_2 - C_1 = \{y - x : x \in C_1, y \in C_2\}$$

contains an interval if

$$\dim_{\mathrm{H}} C_1 + \dim_{\mathrm{H}} C_2 > 1, \tag{1}$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

In 2001, De Moreira and Yoccoz ([7]) proved the conjecture for generic dynamically generated *non-linear* Cantor sets. The conjecture has not been proven for generic linear Cantor sets.

In 1990, Per Larsson put the problem into a probabilistic context in [5], (see also [6]). He considered a very special family of two parameters random Cantor set and proved the conjecture for certain parameter set. Although the main idea of Larsson's argument is brilliant, unfortunately the proof contains significant gaps and incorrect reasonings. In 2011, the present authors gave precise proof for the Larsson's family in [2]. We briefly recall the Larsson's family from [2]: let

$$a > \frac{1}{4} \quad \text{and} \quad 3a + 2b < 1. \tag{2}$$

The first condition is a growth condition, and since

$$\dim_{\mathrm{H}} C_{a,b} = -\frac{\log 2}{\log a},$$

this condition is equivalent to $\dim_{\mathrm{H}} C_{a,b} > 1/2$, which is equivalent to (1).

Larsson's construction is as follows (see also Figure 1): first remove the interval $\left[\frac{1}{2} - \frac{a}{2}, \frac{1}{2} + \frac{a}{2}\right]$ from middle of $[0, 1]$, then the b parts from both the beginning and the end of the unit interval. Then put intervals of length a according to a uniform distribution in the remaining two gaps $\left[b, \frac{1}{2} - \frac{a}{2}\right]$ and $\left[\frac{1}{2} + \frac{a}{2}, 1 - b\right]$. These two randomly chosen intervals of length a are called the level one intervals of the random Cantor set $C_{a,b}$. We write $C^1_{a,b}$ for their union. In both of the two level one intervals we repeat the same construction independently of each other and of the previous step. In this way we obtain four disjoint intervals of length a^2. We emphasize that, because of independence, the relative positions of these second level intervals in the first level ones are in general completely different. Similarly, we construct the 2^n level n intervals of length a^n. We call their union $C^n_{a,b}$. Then Larsson's random Cantor set is defined by

$$C_{a,b} := \bigcap_{n=1}^{\infty} C^n_{a,b}.$$

In this paper, we prove that the conjecture by Pallis and Takens is true for a very broad class of *random* linear Cantor sets. Our result is valid for the Larsson family as well.

Below, we describe the the Cantor sets we investigate in this paper. We throw $m \geq 2$ disjoint intervals $I_1, \dots, I_m$ onto the $I := [0, 1]$ such that the length $r_1, \dots, r_m$ of these intervals are fixed and the left endpoints $D_1 \leq \cdots \leq D_m$ form a random vector $(D_1, \dots, D_m)$ having absolute continuous marginals. This gives the first approximation of our random Cantor set. To get the second approximation we repeat the same process independently in every I_k instead of I. This results in the level two intervals $\{I_{ij}\}^m_{i,j=1}$. Then we repeat the same process independently in all intervals I_{ij} and independently of the previous step. Note that because of the independence the relative positions of the second level intervals are independent of the relative positions of the first level intervals. The intersection of the union of all level-n intervals is a random Cantor set. As we see in Section 1.1 this random Cantor set is the attractor of a random IFS, let us call it $\mathcal{F}$, which consists of linear contractions with deterministic contraction ratios and random translations. The object of our study in this paper is a slightly more general random Cantor set $\mathcal{C}_{\mathcal{F}}$ which is the attractor of a random IFS in which additionally reflections are allowed. For the precise definition see Definition 1.

Our main result is a generalization of [2]:

Main Theorem. *Let $\mathcal{F}$ be a random IFS with similarity dimension larger than $\frac{1}{2}$ informally described above and precisely defined in Definition 1. Let C_1 and C_2 be two*

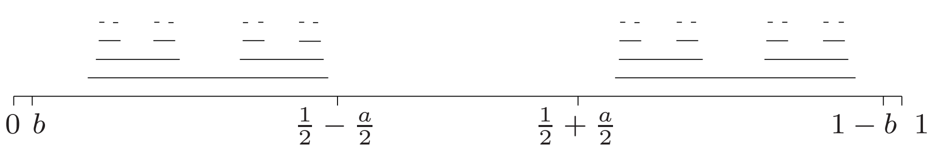

Figure 1: The construction of the Cantor set $C_{a,b}$. The figure shows $C^1_{a,b}, \dots, C^4_{a,b}$. (This figure is taken from [2].)

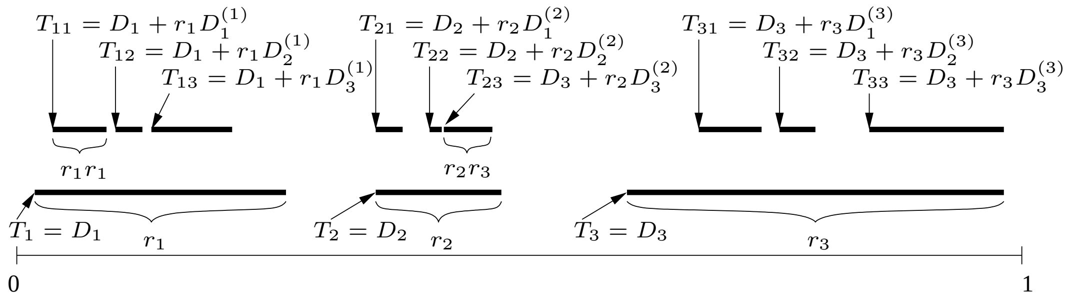

Figure 2: Level 1 and 2 cylinder intervals of our Cantor set when $m = 3$. The randomly chosen left endpoints T_i and T_{ij}, $i, j = 1, 2, 3$ are depicted.

independent copies of the attractor $C_{\mathcal{F}}$. Then

$$C_2 - C_1 \text{ contains an interval a.s.}$$

The essential part of the proof of this theorem is completely different of that of the main result in [2]. The proof in [2] was tailored for the Larsson's family, it does not have the potential to any generalization. The proof presented in this paper is based on a method introduced in [8] in the sense that [8] determines the successive steps that have to be proved so that in the end we get an interval in the algebraic difference almost surely. However, the proofs of the individual steps in this paper are again completely different of the ones in [8].

The cornerstone of the proofs, Main Lemma both in [2] and in this paper are similar to each other in its role. In both papers Main Lemma serves as a starting step of the essential part of the proof. Informally, Main Lemmas state that the associated branching processes are uniformly supercritical, where uniformity is meant in the type of the ancestor. The proof of this property is quite technical in both paper and this is the point in which the proofs follow the same trace. In spite of this similarity, since the result in this paper is more general the proof here is significantly different of the one in [2].

The reminder of the paper consists of three parts. First, in Section 2, we prove that our Main Theorem provided we know the statement of the Main Theorem for intervals having the same length $r_1 = \cdots = r_m = a$. In Section 3, we introduce the key notions and the associated branching process that are needed to the rest of the paper, and we present the idea of the proofs of the rest of the paper in very informal way. The second part, Section 4 contains the essential part of the proof, that is, once we know the uniform super criticality of the associated branching process (this is the Main Lemma), we prove that there exists an interval in the difference almost surely. The third part, Section 5 and 6, contains the proof of the Main Lemma.

1.1 The formal description of our Random Cantor set

First we define the Random Iterated Functions System (RIFS) $\mathcal{F}$ whose attractor $C_{\mathcal{F}}$ is the random Cantor set described above.

Definition 1 (RIFS). *Let*

$$\mathcal{F} = \{f_i(x) = r_i x + D_i\}_{i=1}^{m}. \tag{3}$$

The linear parts $r_1, \dots, r_m \in (-1,1) \setminus \{0\}$ are deterministic. About the random translations $(D_1, \dots, D_m)$, of the functions in $\mathcal{F}$ in (3), we assume the following:

(A1): *$(D_1, \dots, D_m)$ is an m dimensional random variable such that for any $i = 1, \dots, m$, the random variable D_i is absolute continuous w.r.t. the Lebesgue measure.*

(A2): *For any $i, j \in \{1, \dots, m\}$ and $i \neq j$ we have*

$$f_i(0,1) \cap f_j(0,1) = \emptyset.$$

To define the random translations of the iterates of this system we introduce

$$\left\{\left(D_1^{(\mathbf{i})}, \dots, D_m^{(\mathbf{i})}\right)\right\}_{\mathbf{i} \in \Sigma^* \cup \{\emptyset\}}$$

as a set of i.i.d. random variables having the same distribution as that of $(D_1, \dots, D_m)$. Now, the iterates $f_{\mathbf{i}}$ for $\mathbf{i} \in (1, \dots, m)^n$ are defined as follows:

$$\begin{aligned} f_{\mathbf{i}}(x) &= f_{i_1} \circ \cdots \circ f_{i_n}(x) \\ &= r_{i_1}\left(r_{i_2}\left(\dots\left(r_{i_{n-1}}(r_{i_n}x + D_{i_n}^{(i_1 \dots i_{n-1})}) + D_{i_{n-1}}^{(i_1 \dots i_{n-2})}\right)\dots\right) + D_{i_2}^{(i_1)}\right) + D_{i_1}^{(\emptyset)} \\ &= r_{\mathbf{i}}x + T_{\mathbf{i}} \end{aligned} \tag{4}$$

where $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_n}$ and

$$T_{\mathbf{i}} = D_{i_1}^{(\emptyset)} + r_{i_1} D_{i_2}^{(i_1)} + r_{i_1} r_{i_2} D_{i_3}^{(i_1 i_2)} + \dots + r_{i_1} \cdots r_{i_1 \dots i_{n-1}} D_{i_n}^{(i_1 \dots i_{n-1})}. \tag{5}$$

We write $C_{\mathcal{F}}$ for the attractor of the IFS above.

The dimension theory of the RIFS described above is well understood.

The following fact is a direct consequence of the geometric construction presented in [1].

Fact 1 (Dimension of a RIFS). *Let $\mathcal{F}$ be a RIFS and let $s(\mathcal{F})$ denote the solution the equation*

$$\sum_{i=1}^{m} |r_i|^s = 1. \tag{6}$$

Then we have

$$\dim_{\mathrm{H}} C_{\mathcal{F}} = \overline{\dim}_B C_{\mathcal{F}} = \underline{\dim}_B C_{\mathcal{F}} = s(\mathcal{F}) \quad \textit{for } \textbf{\textit{all}} \textit{ realisations.}$$

With these definitions we can state our Main Theorem precisely:

Theorem 1 (Main Theorem). *Let C_1, C_2 be two independent copies of the attractor $C_{\mathcal{F}}$. If $s = s(\mathcal{F}) > \frac{1}{2}$ (that is $\dim_{\mathrm{H}} C_{\mathcal{F}} > \frac{1}{2}$), then $C_2 - C_1$ contains an interval a.s. . On the other hand, if $s < \frac{1}{2}$ then $\dim_{\mathrm{H}}(C_2 - C_1) < 1$, so $C_2 - C_1$ cannot contain any intervals.*

2 The proof of the main result

Homogeneous random IFS are the ones where all the contraction ratios are the same (see precise definition below). In this section we verify that it is enough to prove our main theorem for the homogeneous RIFS.

Definition 2 (Generated RIFS)**.** *Let $\mathcal{F} = \{f_i\}_{i=1}^m$ be an RIFS. We say that $\mathcal{G}$ is a generated RIFS of $\mathcal{F}$ if any element of $\mathcal{G}$ is a composition of some elements of $\mathcal{F}$. More precisely, $\mathcal{G}$ is of the form of*

$$\mathcal{G} = \{f_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{I}},$$

where $\mathcal{I} \subset \{1, \dots, m\}^n$, for some n and $\mathcal{I}$. (We do not require here that $\mathcal{I}$ is a cut set.)

Now we prove that a generated RIFS is always a RIFS that is it satisfies **(A1)** and **(A2)**.

Lemma 1. *Let $\mathcal{F}$ be an RIFS and let $\mathcal{G}$ be an arbitrary generated RIFS of $\mathcal{F}$. Then $\mathcal{G}$ itself is an RIFS.*

Proof. The only fact to check is that for all $\mathbf{i} \in \mathcal{I}$ the random translation $T_{\mathbf{i}}$ is absolute continuous. However, using (5), this follows from the fact that the random variables $D_{i_1}^{(\emptyset)}, D_{i_2}^{(i_1)}, D_{i_3}^{(i_1 i_2)}, \dots, D_{i_n}^{(i_1 \dots i_{n-1})}$ are absolute continuous random variables and they are independent of each other. □

Definition 3 (Homogeneous RIFS)**.** *We say that $\mathcal{H}$ is a homogeneous RIFS if $\mathcal{H}$ satisfies the conditions of Definition 1 and all the contraction ratios are equal to the same* positive *number a. We write $C_{\mathcal{H}}$ for the attractor of the RIFS above.*

Since in the rest of the paper we work mostly with homogeneous RIFS we summarize here their most important properties:

Remark 1. *Let $\mathcal{H}$ be a homogeneous RIFS. Then $\mathcal{H}$ is of the form*

$$\mathcal{H} = \{ax + U_i\}_{i=1}^K,$$

where $K \geq 2$, $a > 0$ and the random translations $(U_1, \dots, U_K)$ satisfy:

(H1) *For any $i = 1, \dots, K$, U_i is absolute continuous w.r.t. the Lebesgue measure.*

(H2) *For any $i = 1, \dots, K-1$ we have*

$$0 \leq a + U_i \leq U_{i+1} \leq 1 - a.$$

Note: since $s(\mathcal{H}) = -\frac{\log K}{\log a}$ for a homogeneous RIFS $\mathcal{H}$ the condition $\dim_{\mathrm{H}} C_{\mathcal{H}} > \frac{1}{2}$ translates to $\dfrac{1}{K^2} < a < \dfrac{1}{K}$.

Our main result for homogeneous RIFS states that:

Theorem 2. *Let $\mathcal{H}$ be a homogeneous RIFS satisfying $a > \frac{1}{K^2}$ (that is $s(\mathcal{H}) > \frac{1}{2}$ and $\dim_{\mathrm{H}}(C_{\mathcal{H}}) > \frac{1}{2}$). Let C_1 and C_2 two independent copies of $C_{\mathcal{H}}$. Then $C_2 - C_1$ contains an interval a.s.. On the other hand if $a < \frac{1}{K^2}$ then $C_2 - C_1$ cannot contain any intervals.*

Our aim is to prove that Theorem 1 follows from Theorem 2. In order to do that we need a lemma which is a randomized version of [10, Proposition 6]:

Lemma 2. *Let $\mathcal{F}$ be a RIFS. Then for any $\delta > 0$ there exists a generated homogeneous RIFS $\mathcal{H}$ such that,*

$$C_{\mathcal{F}} \supset C_{\mathcal{H}} \quad \textit{and} \quad \dim_{\mathrm{H}} C_{\mathcal{H}} > \dim_{\mathrm{H}} C_{\mathcal{F}} - \delta \tag{7}$$

Proof. Using the same argument as in the first paragraph of the proof of [10, Proposition 6] one can easily see that we can find a generated RIFS

$$\mathcal{F}' = \{\varphi_i(x) = q_i x + T_i\}_{i=1}^{M} \tag{8}$$

satisfying:

(b1) $\dim_{\mathrm{H}}(C_{\mathcal{F}'}) > \dim_{\mathrm{H}}(C_{\mathcal{F}}) - \delta/2$

(b2) $q_i > 0$ for all $i = 1, \dots, M$.

Although the proof of [10, Proposition 6] is presented for IFS with deterministic translations it can be easily extended to random IFSs in Definition 1 because it operates only with the contraction ratios independently of the size of the translations.

We define $\mathcal{H}$ as a proper subsystem of $\mathcal{F}'$ following the steps of the proof of [10, Proposition 6]. According to Lemma 1 $\mathcal{H}$ is a RIFS as well. So we have to prove that the dimension of $\mathcal{H}$ is close enough to the dimension of $\mathcal{F}'$ in the sense of (7).

We only repeat the key steps of the proof of [10, Proposition 6]. Namely, for any k let us consider the simplex

$$\Delta_k = \left\{ \mathbf{x} \in \mathbb{R}^M : x_i \geq 0, \sum_{i=1}^{M} x_i = k \right\}.$$

Clearly, for $p_i := q_i^{s(\mathcal{F}')}$ we have $\sum_{i=1}^{M} k p_i \mathbf{e}_i \in \Delta_k$. We define $\mathbf{v}(k) \in \Delta_k$ as an all integer coordinate element of Δ_k of minimal distance from $\sum_{i=1}^{M} k p_i \mathbf{e}_i$. Let

$$\mathcal{I}_k := \left\{ \mathbf{i} = (i_1, \dots i_k) : \sum_{\ell=1}^{k} \mathbf{e}_{i_\ell} = \mathbf{v}(k) \right\}.$$

Now we define

$$\mathcal{H}_k := \{\varphi_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{I}_k}.$$

By the definition of $\mathcal{I}$, for every $\varphi_{\mathbf{i}} \in \mathcal{H}_k$ the contraction ratio is the same constant $\rho(k)$, where $q_{\mathbf{i}} = \prod_{\ell=1}^{k} q_{i_\ell} = \prod_{i=1}^{M} q_i^{v_i(k)} =: \rho(k)$.

Fact 1 implies that for all realizations

$$\dim_{\mathrm{H}} \mathcal{F}' = s(\mathcal{F}') \quad \text{and} \quad \dim_{\mathrm{H}} \mathcal{H} = s(\mathcal{H}) = -\frac{\log \#\mathcal{H}_k}{\log \rho(k)}.$$

Using this fact, it follows from the proof of [10, Proposition 6] that

$$\lim_{k\to\infty} \dim_{\mathrm{H}}(\mathcal{H}) = \dim_{\mathrm{H}}(\mathcal{F}').$$

Let us fix a k such that

$$\dim_{\mathrm{H}}(\mathcal{H}) > \dim_{\mathrm{H}}(\mathcal{F}') - \frac{\delta}{2}.$$

Let $\mathcal{H} := \mathcal{H}_k$ and $a := \rho(k)$. Hence using (b1), we obtain that

$$\dim_{\mathrm{H}}(\mathcal{H}) > \dim_{\mathrm{H}}(\mathcal{F}') - \frac{\delta}{2} > \dim_{\mathrm{H}}(\mathcal{F}) - \delta.$$

This completes the proof of the lemma. $\square$

Now we are ready to prove our main Theorem assuming that Theorem 2 holds.

Proof of Theorem 1 assuming Theorem 2. Given a RIFS $\mathcal{F}$. Put $\delta := s(\mathcal{F}) - 1/2$. We consider the homogeneous RIFS $\mathcal{H}$ constructed in Lemma 2. By the choice $\delta = s(\mathcal{F}) - 1/2$, $s(\mathcal{H}) > \frac{1}{2}$ holds, so for $\mathcal{H}$ Theorem 2 can be applied.

Let $C_1^{(\mathcal{F})}$ and $C_2^{(\mathcal{F})}$ be two independent copies of $C_{\mathcal{F}}$. Further, let $C_1^{(\mathcal{H})} \subset C_1^{(\mathcal{F})}$ and $C_2^{(\mathcal{H})} \subset C_2^{(\mathcal{F})}$ be independent attractors of $\mathcal{H}$. Since $C_2^{(\mathcal{H})} - C_1^{(\mathcal{H})}$ contains an interval almost surely (by Theorem 2) the larger set $C_2^{(\mathcal{F})} - C_1^{(\mathcal{F})}$ also contains an interval almost surely. $\square$

3 The idea of the proof of Theorem 2 and the Main Lemma

The proof of Theorem 2 consists of two parts. The first part of the proof is presented in Section 4. The second part, the proof of Main Lemma, which is more technical, will be presented in Sections 5 and 6. Below, we first introduce a renormalization operator Φ in order to present the idea of the proof.

3.1 The renormalization operator Φ

The following simple observation allows us to present the problem in geometrically more understandable way. Let us denote by $Proj_{45^\circ}$ the 45° projection of $\mathbb{R}^2$ to the vertical axes, that is, for $(x, y) \in \mathbb{R}^2$

$$Proj_{45^\circ}(x, y) = y - x.$$

Now, observe that

$$x \in C_2 - C_1 \iff x \in Proj_{45^\circ}(C_1 \times C_2) \iff e(x) \cap C_1 \times C_2 \neq \emptyset,$$

where $e(x)$ denotes the 45° line through $(0, x)$. Remark that since $C_1, C_2 \subset [0,1]$ we have

$$Proj_{45^\circ}(C_1 \times C_2) \subset [-1, 1].$$

By the construction of the Cantor sets C_1, C_2, for $x \in [-1,1]$ we have

$$e(x) \cap (C_1 \times C_2) \neq \emptyset \iff \forall n: \ e(x) \cap (C_1^n \times C_2^n) \neq \emptyset \iff \text{for arbitrary subsequence } (n_k): \ e(x) \cap (C_1^{n_k} \times C_2^{n_k}) \neq \emptyset,$$

where C_1^n and C_2^n denote the level-n cylinder intervals of Cantor sets C_1 and C_2 respectively. Hence $C_1^n \times C_2^n$ denotes the collection of level-n cylinder squares.

The "type" of the intersection of a square and the line $e(x)$ has to be also investigated since it is not enough registering the fact of the intersection. Later, the "type" defined below enables us to use branching process techniques. So, let Q be a level-n square. More precisely, if (u, v) denotes the left bottom corner of a level-n square Q, that is, $Q = [u, u + a^n] \times [v, v + a^n]$ then we define the **intersection type** of Q with the line $e(x)$ as follows:

$$\Phi(Q, x) := \begin{cases} \frac{u-v+x}{a^n} & \text{if } e(x) \text{ intersects } Q \\ \Theta & \text{otherwise,} \end{cases} \tag{9}$$

where the symbol Θ represents the emptiness of the intersection and we call Φ the renormalization operator. See Figure 3. Observe that if the intersection is non-empty then the

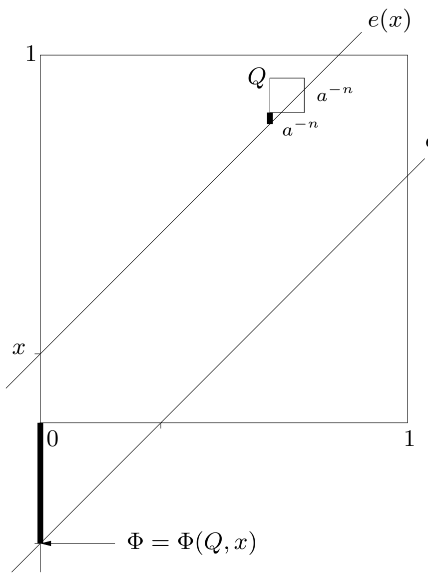

Figure 3: $\Phi(Q, x)$ is one of the n-th generation offspring of x determined by a level-n cylinder square Q.

signed length of the intersection Φ is rescaled into the interval $[-1,1]$. Further, observe that $\Phi(Q,x)>0$ if and only if the center of Q is located below the line $e(x)$ and $e(x)$ meets Q.

3.2 Idea of the proof of Theorem 2

Now, we are ready to present the idea of the proof of Theorem 2 in a very informal way.
►*For an appropriately chosen collection of closed subintervals $H_2 \subset [-1,1]$ and small fixed length $h(H_2)>0$ and positive integer $r(H_2)$ we have the following:*
Let $\xi>0$ be an arbitrarily small number. Then, there exists a positive integer $n(\xi)$ such that with probability at least $1-\xi$ we can find an interval $J \subset [-1,1]$ of length $h(H_2)$ such that for any $x \in J$ for every $M \in \mathbb{N}$ there are exponentially many level $n_M = n(\xi) + M \cdot r(H_2)$ squares $Q_1^{(M)}, Q_2^{(M)} \dots$ such that $\Phi(Q_i^{(M)}, x) \in H_2$.

One of the main ingredients of Theorem 2 is verified in the Main Lemma, in Sections 5 and 6 performing a rather technical proof. Proving the Main Lemma is the initial step in the machinery introduced in [8]. At the present state of the formalism it is easier to formulate a consequence of the Main Lemma, Corollary 1 on page 15. It states the following. *One can find pairs of appropriately large collection of closed subintervals $H_1, H_2 \subset [-1,1]$ such that $H_1 \subset \mathrm{int} H_2$ and for any $x \in H_2$ the expected number of level-1 squares $Q_1, Q_2, \dots$ such that $\Phi(Q_i, x) \in H_1$ is bigger than 1 uniformly in x.*

In the rest of this section we will put the problem in branching process context and formulate the Main Lemma.

3.3 The probability space of the squares

Let $\mathcal{T}$ be the K array tree which is the set of finite words over the alphabet $\{1,\dots,K\}$. Following the definition of $C_{\mathcal{F}}$ in Definition 1, let the probability space of our Cantor set be

$$\Omega_1 = \bigotimes_{\mathbf{i} \in \mathcal{T}} \operatorname{supp} U_{\ell(\mathbf{i})}^{\mathbf{i}^*},$$

where $\ell(\mathbf{i})$ denotes the last coordinate of $\mathbf{i}$ and $\mathbf{i}^*$ stands for the vector that is built up from the all but last coordinate of $\mathbf{i}$, that is, if $\mathbf{i} = (i_1 \dots i_n)$ then $\mathbf{i}^* = (i_1 \dots i_{n-1})$. The corresponding σ-algebra is $\mathcal{B}_1$ is the generated Borel σ-algebra. The probability measure of our Cantor set is

$$\mathbb{P}_1 = \delta_0 \times \prod_{\mathbf{i} \in \mathcal{T}} d\left(U_{\ell(\mathbf{i})}^{\mathbf{i}^*}\right),$$

where $d()$ denotes the probability distribution of the random variable in the parenthesis and δ_0 is the Dirac mass at 0 associated with the mass at the root of $\mathcal{T}$. So the probability space for $C_1 \times C_2$ is as follows:

$$\Omega := \Omega_1 \times \Omega_1, \qquad \mathcal{B} := \mathcal{B}_1 \times \mathcal{B}_1, \qquad \mathbb{P} := \mathbb{P}_1 \times \mathbb{P}_1. \tag{10}$$

An element of Ω is a pair of labeled K array trees. The $(K^2)^n$ level-n pairs of indices $(i_1 i_2 \dots i_n, j_1 j_2 \dots j_n)$ are naturally associated with level-n squares $Q_{i_1 i_2 \dots i_n, j_1 j_2 \dots j_n} =$

$I^{(1)}_{i_1 i_2 \dots i_n} \times I^{(2)}_{j_1 j_2 \dots j_n}$ of size $a^n \times a^n$, see Figure 4. The indexing of the squares is inherited from the indexing of the cylinder sets of the Cantor sets C_1 and C_2 hence it follows "antimatrix" numbering.

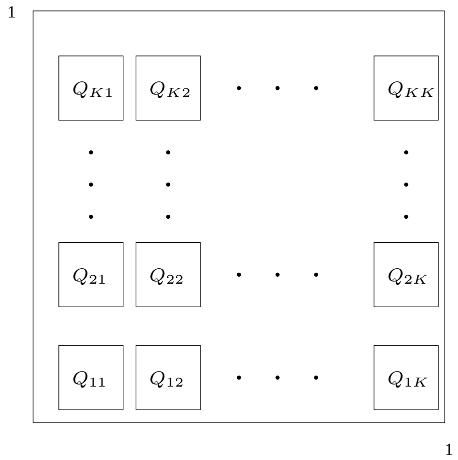

Figure 4: The level-1 squares Q_{ij}, $i, j = 1, \dots, K$.

3.4 The branching process

In the introduction of our multi-type branching process we follow [2, Section 3.3] since the branching process constructed in this paper is similar to the one presented in [2].

On the probability space Ω we define a multi type branching process $\mathcal{Z} = (\mathcal{Z}_n)_{n=0}^{\infty}$. The type space T is a compact subset of $[-1, 1]$, for the moment think of $T = [-1, 1]$.

The types of the descendants of an individual $x \in [-1, 1]$ are the intersection types of the level-1 squares with line $e(x)$ as defined in (9). More precisely, let $\mathcal{Z}_0 = x$ and let $Z_{ij} := \Phi(Q_{i,j}, x)$ then

$$\mathcal{Z}_1 = \{Z_{i,j} : i, j = 1 \dots, K\}$$

Note that although we speak of Θ as a type, it is *not* an element of T.

Therefore, the level-n children of ancestor $x \in T$ are the signed length of the rescaled intersections of the level-n squares with the line $e(x)$. More precisely, let $\mathcal{Z}_0 = \{x\}$ then for any $n \geq 1$

$$\mathcal{Z}_n = \{\Phi(x, Q_{\mathbf{t}}) : \mathbf{t} \in \mathcal{T}_n \times \mathcal{T}_n,\ e(x) \cap Q_{\mathbf{t}} \neq \emptyset\},$$

where $\mathcal{T}_n := \{1, \dots, K\}^n$.

We remark that the process $(\mathcal{Z}_n)_{n=0}^{\infty}$ is a Markov chain since an individual in $\mathcal{Z}_n$ give birth descendants independently of the individuals of the same generation if $\mathcal{Z}_{n-1}$ is given.

A collection of squares all with type Θ is an absorbing state: it only generates squares with type Θ. This is obvious from the definition of $\Phi(Q, x)$, but we will extend this property to the case of smaller type spaces T, where by definition a square has type Θ if its type is not in T (this will be further explained in Section 5).

For a Borel set $A \subset T$ the natural number $\mathcal{Z}_n(A)$ represents the number of objects in generation n whose type falls into the set A.

A major role in our analysis is played by the expectations $\mathbf{E}_x[\mathcal{Z}_n(A)]$, for $A \subset T$, $n \geq 1$. Since

$$\mathcal{Z}_n(A) = \sum_{\mathbf{t} \in \mathcal{T}_n \times \mathcal{T}_n,\ e(x) \cap Q_{\mathbf{t}} \neq \emptyset} \mathbf{1}\{\Phi(x, Q_{\mathbf{t}}) \in A\}$$

for $n = 1$ we have

$$\begin{aligned} \mathbf{E}_x[\mathcal{Z}_1(A)] &= \int_\Omega \mathcal{Z}_1(A) \, \mathrm{d}\mathbb{P}_x = \int_\Omega \sum_{i,j=1}^K \mathbf{1}\{\Phi(x, Q_{ij}) \in A\} \, \mathrm{d}\mathbb{P}_x \\ &= \sum_{i,j=1}^K \mathbb{P}(\Phi(Q_{ij}, x) \in A) = \sum_{i,j=1}^K \int_A \phi_{ij}(x, y) \, \mathrm{d}y, \end{aligned} \tag{11}$$

where $\phi_{ij}(x, \cdot)$ denotes the density function of the random variable $\Phi(Q_{ij}, x)$ (apart from an atom in Θ) for $i, j = 1, \ldots, K$. In Section 5 these densities will be determined explicitly. It follows that for $n = 1$

$$M_n(x, A) := \mathbf{E}_x[\mathcal{Z}_n(A)]$$

has a density $m_1(x, y)$, called the kernel of the branching process, given by

$$m(x, y) := m_1(x, y) = \sum_{i,j=1}^K \phi_{ij}(x, y). \tag{12}$$

We remark that if M_1 has a density then M_n also has a density. Let us write $m_n(x, \cdot)$ for the density of $M_n(x, \cdot)$. The branching structure of $\mathcal{Z}$ yields (see [3, p.67])

$$m_{n+1}(x, y) = \int_T m_n(x, z) m_1(z, y) \, \mathrm{d}z. \tag{13}$$

One of the main problem in the argument is finding the proper type space $T \subset [-1, 1]$. In Section ... we will prove that it can be constructed a type space T such that Theorem 3 holds and condition (**C2**) below also satisfies. Further, $m(\cdot, \cdot)$ is continuous on the compact set $T \times T$.

3.5 Supercritical branching process with uniformly positive kernel

We will prove in Sections 5 and 6 that there exists an integer n_0 such that m_{n_0} is a uniformly bounded function, that is, there exist $0 < a_{\min} < a_{\max}$ such that for all $x, y \in T$ we have

$$0 < a_{\min} \leq m_{n_0}(x, y) \leq a_{\max} < \infty. \tag{\textbf{C1}}$$

In the next step we consider the following two operators:

$$\begin{aligned} F:\ g(x) &\mapsto \int\limits_T m_1(x,y)\cdot g(y)\ \mathrm{d}y \\ G:\ h(y) &\mapsto \int\limits_T h(x)\cdot m_1(x,y)\ \mathrm{d}x. \end{aligned} \tag{14}$$

We cite the following theorem from [3, Theorem 10.1]:

Theorem 3 (Harris)**.** *It follows from* (**C1**) *that the operators in (14) have a common dominant eigenvalue ρ. Let $\mu(x)$ and $\nu(y)$ be the corresponding eigenfunctions of the first and second operator in (14) respectively. Then the functions $\mu(x)$ and $\nu(y)$ are bounded and uniformly positive. Moreover, apart from a scaling, μ and ν are the only non-negative eigenfunctions of these operators. Further, if we normalize μ and ν so that $\int \mu(x)\nu(x)\,\mathrm{d}x = 1$, which will be henceforth assumed, then for all $x, y \in T$ as $n \to \infty$*

$$\left|\frac{m_n(x,y)}{\rho^n} - \mu(x)\nu(y)\right| \leq C_1\,\mu(x)\nu(y)\Delta^n,$$

where the bound $\Delta < 1$ can be taken independently of x and y, and the constant C_1 is independent of x, y and n.

In Sections 5 and 6 we will prove our Main Lemma that we state now.

Main Lemma. *Let us be given a homogeneous RIFS $\mathcal{H}$. Then we can find a type space T such that condition (**C1**) is satisfied, and hence Theorem 3 holds. Moreover, the Perron-Frobenius eigenvalue ρ is greater than one:*

$$\rho > 1. \tag{\textbf{C2}}$$

Putting $\delta = \rho - 1 > 0$ we can find a continuous function $f : T \to (0, \infty)$ such that

$$Ff(x) = (1+\delta)f(x), \tag{15}$$

where the operator F was defined in (14).

Note that f is uniformly positive and bounded function because T is compact and f continuous and positive on T.

We remark that Main Lemma states that with the type-space constructed in Sections 5 and 6 and the types of the descendants given in Section 3.4, the associated branching process is supercritical.

We also note that (15) is equivalent to the following:

$$\forall x \in T: \quad \mathbf{E}_x f\left(\mathcal{Z}_1(T)\right) = (1+\delta)f(x).$$

4 The proof of Theorem 2 assuming Main Lemma

Let $T(0)$ be the set (union of finite open subintervals of $(-1,1)$) that we construct in Sections 5 and 6. In Theorem 4, it turns out that there exists an interval $(0,\varepsilon_0)$ such that for any $\varepsilon \in (0,\varepsilon_0)$ the compact set $T = T(\varepsilon) := T(0) \setminus B(\partial T, \varepsilon) \neq \emptyset$ is a type-space for which Main Lemma is true, where for any set H and radius $r > 0$ we used the notation $B(H,r) := \bigcup\{(x-r, x+r) : x \in H\}$.

Fix an $\varepsilon \in (0,\varepsilon_0)$ and let $\eta_0 \in (0, \varepsilon_0 - \varepsilon)$.

Using Theorem 4 (Main Lemma) we prove the following.

Lemma 3. *There exist $\eta \in (0,\eta_0]$ such that introducing the notations*

$$g_1(x) = f(x) \cdot \mathbf{1}_{T(\varepsilon+\eta)}(x), \quad x \in T(\varepsilon)$$

we have

$$Fg_1(x) > \left(1 + \frac{\delta}{2}\right) f(x) \ \text{ for any } \ x \in T(\varepsilon).$$

Proof. What we need to show it is that we can find $\eta \in (0,\eta_0]$ such that

$$\forall x \in T(\varepsilon) \quad \int_{T(\varepsilon+\eta)} m(x,y) f(y) \,\mathrm{d}y > \left(1 + \frac{\delta}{2}\right) f(x). \tag{16}$$

For $\eta \in [0,\eta_0]$ and $x \in T(\varepsilon)$ we define

$$G(x,\eta) := \int_{T(\varepsilon+\eta)} m(x,y) f(y) \,\mathrm{d}y.$$

Using (15) we have $G(x,0) = (1+\delta) f(x)$. Hence (16) is equivalent to the following:

$$\forall x \in T(\varepsilon), \quad G(x,\eta) > G(x,0) - \frac{\delta}{2} f(x).$$

Hence for showing (16) it is enough to prove that there exists $\eta \in (0,\eta_0]$ small enough such that

$$\sup_{x \in T(\varepsilon)} [G(x,0) - G(x,\eta)] < \frac{\delta}{2} \min_{x \in T(\varepsilon)} f(x). \tag{17}$$

Since $G(x,\eta)$ is continuous on the compact set $T(\varepsilon) \times [0,\eta_0]$ we have that it is uniformly continuous on $T(\varepsilon) \times [0,\eta_0]$. Therefore, one can find $\eta \in (0,\eta_0]$ small enough such that (17) holds.

□

Let us define

$$H_1 := T(\varepsilon + \eta), \qquad H_2 := T(\varepsilon).$$

For $x \in H_2$ let $\mathcal{A}_n(x, H_1)$ be the set of types of those level n descendants of x which fall in H_1. Let

$$A_n(x, H_1) = \#\mathcal{A}_n(x, H_1).$$

Concerning the associated multi-type branching process, an easy consequence of Lemma 3 is the following.

Corollary 1. *For any $n \geq 1$ we have*

$$\forall x \in H_2 \quad F^n \mathbf{1}_{H_1}(x) > \left(1 + \frac{\delta}{4}\right)^n \frac{\min_{H_2} f(x)}{\max_{H_1} g_1(x)} \cdot \mathbf{1}_{H_2}(x) \ .$$

Consequently, there exist a positive integer r such that

$$\forall x \in H_2 \quad F^r \mathbf{1}_{H_1}(x) > 6 \cdot \mathbf{1}_{H_2}(x). \tag{18}$$

That is

$$\forall x \in H_2, \mathbf{E} A_r(x, H_1) > 6 \cdot \mathbf{1}_{H_2}(x). \tag{19}$$

The following statement, Lemma 5 is a slight generalization of the easy part of the Cramér theorem.

Let F_x denote the probability distribution function of $A_r(x, H_1)$ for $x \in H_2$.

Let C be a positive integer and let $x_1, \ldots, x_C \in H_2$ be an arbitrary sequence. Let $Z_{x_1}^{(1)}, \ldots, Z_{x_C}^{(C)}$ be a sequence of independent random variables such that the distribution function of $Z_{x_i}^{(i)}$ is F_{x_i}. We will prove the following.

Our next lemma is a corollary of the Hoeffding inequality[4]:

Lemma 4 (Hoeffding)**.** *Assume $Y_1, \ldots, Y_C$ are independent random variables such that for any $i = 1, \ldots, C$ we have $a_i \leq Y_i \leq b_i$ for some real numbers a_i, b_i. Let $S_C = \sum_{i=1}^{C} Y_i$ and let t be a positive real number then we have*

$$\mathbb{P}(S_C - \mathbf{E} S_C > t) \leq \exp\left\{ - \frac{2t^2}{\sum_{i=1}^{C} (b_i - a_i)^2} \right\}.$$

Lemma 5. *There exists $0 < \tau < 1$ such that for $C \geq 1$*

$$\mathbb{P}\left(Z_{x_1}^{(1)} + \cdots + Z_{x_C}^{(C)} < 4C\right) \leq \tau^C.$$

Proof. Let $m_{x_i} := \mathbf{E} Z_{x_i}^{(i)}$ and in general $m_x := \mathbf{E} Z_x$. We have the following chain of equalities:

$$\begin{aligned} \mathbb{P}\left(Z_{x_1}^{(1)} + \cdots + Z_{x_C}^{(C)} < 4C\right) &= \mathbb{P}\left(Z_{x_1}^{(1)} - m_{x_1} + \cdots + Z_{x_C}^{(C)} - m_{x_C} < 4C - \sum_{i=1}^{C} m_{x_i} \right) \\ &= \mathbb{P}\left(m_{x_1} - Z_{x_1}^{(1)} + \cdots + m_{x_C} - Z_{x_C}^{(C)} > \sum_{i=1}^{C} (m_{x_i} - 4) \right) \end{aligned}$$

Now, we want to apply Hoeffding inequality for

$$Y_i = m_{x_i} - Z_{x_i}^{(i)}, \quad S_C = \sum_{i=1}^{C} Y_i, \text{ and } t = \sum_{i=1}^{C} (m_{x_i} - 4).$$

First, note that by the definition of r in (19) we have $m_x > 6$ for any $x \in H_2$. Therefore, $t > 0$. Further, $\mathbf{E} S_C = 0$.

Since each line $e(x)$ can intersect at most $2 \cdot K^r$ level-r squares we have

$$0 \leq Z_x \leq 2 \cdot K^r \text{ and } m_{x_i} - 2 \cdot K^r \leq Y_i \leq m_{x_i}.$$

Thus, we have

$$\mathbb{P}\left(m_{x_1} - Z_{x_1}^{(1)} + \cdots + m_{x_C} - Z_{x_C}^{(C)} > \sum_{i=1}^{C} (m_{x_i} - C) \right) \leq$$
$$\exp\left\{ - \frac{2\left(\sum_{i=1}^{C}(m_{x_i} - 4)\right)^2}{\sum_{i=1}^{C}(2 \cdot K^r)^2} \right\} \leq$$
$$\exp\left\{ - \frac{2\left(\sum_{i=1}^{C}(6 - 4)\right)^2}{C \cdot 4 \cdot K^{2r}} \right\} = \exp\left\{ - \frac{8C^2}{4C \cdot K^{2r}} \right\} = \left(\exp\left\{ - \frac{2}{K^{2r}} \right\} \right)^C,$$

where we used $m_x > 6$. Since $\tau = \exp\left\{-\frac{2}{K^{2r}}\right\} < 1$, this proves the Lemma.

□

Let us denote by c_1 the length of the smallest interval in $T(0)$. Let $n_1 = \left\lceil \log_a \frac{c_1}{|H_1|} \right\rceil$. Further, let ℓ_1 be the length of the smallest interval in H_1.

Lemma 6. *For any fixed $n \geq n_1$ and for any $\omega \in \Omega$ there exists an interval $J = J(\omega) \subset T(0)$ of length $|J| = \ell_1 a^n$ such that for any $x \in J$*

$$A_n(x, H_1)(\omega) \geq \frac{|H_1|}{2}(K^2 a)^n =: N(n).$$

Proof. The support of the kernel will be chosen in such a way that for any $x \in [-1, 1] \backslash T(0)$ and for any $n \geq 1$ we have $\mathbb{P}(A_n(x, T(0)) > 0) = 0$. See Fact 3.

By definition of $A_n(x, H_1)(\omega)$ we have

$$\begin{aligned}\int_{[-1,1]} A_n(x, H_1)(\omega)\, \mathrm{d}x &= \int_{T(0)} A_n(x, H_1)(\omega)\, \mathrm{d}x \\ &= K^{2n}|H_1|a^n = 2N(n),\end{aligned}$$

here, we have integrated K^{2n} characteristic functions of sets of length $|H_1|a^n$.

In this way, also using the initial remark, there exists an $x(\omega) \in T(0)$ such that

$$A_n(x, H_1)(\omega) \geq 2N(n).$$

This implies that there exists an interval $J = J(\omega) \subset T(0)$ of length $|J| = \ell_1 \cdot a^n$ such that for any $x \in J$

$$\forall x \in J \quad A_n(x, H_1)(\omega) \geq N(n),$$

we remind the reader that ℓ_1 was defined before the statement of the Lemma.

□

We partition each interval of $T(0)$ into intervals of equal length. If I is an interval of $T(0)$ then we partition it into $\left\lceil 3\frac{|I|}{\ell_1 a^n}\right\rceil$ subintervals. In this way we obtain a partition of $T(0)$ into the intervals $J_1, \ldots, J_L$ such that for any k, $|J_k| \leq |J|/3$, where J was defined in Lemma 6.

For $\omega \in \Omega$ let $k(\omega) \in \{1, \ldots, L\}$ be chosen such that

$$J_{k(\omega)} \subset J(\omega), \quad J_{k(\omega)-1} \not\subset J(\omega),$$

where $J(\omega)$ is the interval defined in Lemma 6. Let

$$\Omega_l = \{\omega : k(\omega) = l\}$$

Note that, by Lemma 6 we have

$$\Omega = \bigcup_{l=1}^{L} \Omega_l.$$

Let

$$a_k(n) = (K^2)^{n+kr} \tau^{2^{k-1} \cdot N(n)}.$$

For fixed $\xi > 0$ let $n_2 \geq n_1$ be chosen such that for any $n \geq n_2$ and for any $k \geq 0$

$$a_k(n) < \frac{1}{2} \text{ and } \sum_{k=0}^{\infty} a_k(n) < \xi/2.$$

Lemma 7. *Fix an arbitrary $n \geq n_2$ and $l \in \{1, \ldots, L\}$. Then*

$$\forall x \in J_l \text{ and } \forall \omega \in \Omega_l, \quad A_n(x, H_2)(\omega) > N(n). \tag{20}$$

Further,

$$\mathbb{P}\left(A_{n+Mr}(x, H_2) > 2^M \cdot N(n), M = 0, 1, \ldots, \ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{\infty}(1 - a_k(n)). \tag{21}$$

Proof. Equation (20) follows from Lemma 5 and the previous definition.

Concerning (21). Let X_k be a ηa^{2n+kr} dense set in J_l, where η has been set in Lemma 3. X_k can be chosen such that $\#X_k \leq \frac{\ell_1 a^n}{\eta a^{2n+kr}} = \frac{\ell_1}{\eta} a^{-(n+kr)} < (K^2)^{n+kr}$ if $n \geq n_2$.

Using Lemma 5 it can be proved that for any $x \in J_l$

$$\mathbb{P}\left(A_{n+r}(x, H_1) \leq 2N(n) \mid \Omega_l\right) \leq \tau^{N(n)}. \tag{22}$$

Indeed, let

$$C := A_n(x, H_2)(\omega)/2$$

and we define $\widetilde{\mathcal{A}}_n(x, H_2)$ as follows: first we order the elements of $\mathcal{A}_n(x, H_2)$ in the natural way and we choose every second elements to obtain

$$\widetilde{\mathcal{A}}_n(x, H_2) := \{y_1, \ldots, y_C(\omega)\}.$$

Put

$$Y_i := A_r(y_i, H_1), \quad i = 1, \dots, C(\omega).$$

In this way $Y_1, Y_2, \dots, Y_C$ is a sequence of independent random variables. Lemma 5 yields:

$$\mathbb{P}\left(\sum_{i=1}^{C} Y_i < 2N(n) \mid \Omega_l\right) \le \mathbb{P}\left(\sum_{i=1}^{C} Y_i < 4C \mid \Omega_l\right) \le \mathbf{E}(\tau^C \mid \Omega_l) < \tau^{N(n)}$$

since $2C \ge N(n)$ on Ω_l by Lemma 6. On the other hand, by the definition of C and Y_i's

$$\sum_{i=1}^{C} Y_i \le A_{n+r}(x, H_1).$$

Hence equation (22) follows.

To prove equation (21) we will use induction. More precisely, we will prove that the inequality

$$\mathbb{P}\left(A_{n+kr}(x, H_2) > 2^k \cdot N(n), \forall 0 \le k \le M,\ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{M}(1 - a_k(n)). \tag{23}$$

holds for any positive integer M.

For $M = 1$, by (22) we obtain:

$$\mathbb{P}\left(\exists x \in X_1,\ A_{n+r}(x, H_1) \le 2N(n) | \Omega_l\right) \le \# X_1 \cdot \tau^{N(n)}.$$

Recall that X_1 was defined as an ηa^{2n+r}-dense subset of J_l. Using $|X_1| \le (K^2)^{n+r}$ we have

$$\mathbb{P}\left(A_{n+r}(x, H_1) > 2N(n), x \in X_1 \mid \Omega_l\right) \ge 1 - (K^2)^{n+r} \tau^{N(n)}. \tag{24}$$

Next, our purpose is to extend the inequality (24) from all $x \in X_1$ to all $x \in J_l$. Let us fix $k \ge 1$.

We will use the following fact.

Fact 2. *For any $k \ge 1$ and $l = 1, \dots, L$ we have*

$$\left\{A_{n+kr}(x, H_1) > 2^k N(n), x \in X_k\right\} \cap \Omega_l \subset \left\{A_{n+kr}(x, H_2) > 2^k N(n), x \in J_l\right\} \cap \Omega_l, \tag{25}$$

and the event

$$\left\{\forall x \in J_l :\ A_{n+kr}(x, H_2) \ge 2^k N(n)\right\} \tag{26}$$

is measurable

Proof of Fact 2. Using the definition of $H_1 = T(\varepsilon + \eta)$ and $H_2 = T(\varepsilon)$, if for some x' and ω one has $A_{n+kr}(x', H_1)(\omega) > 2^k N(n)$, then for the same ω and for any x such that $|x - x'| < \eta a^{n+kr}$ and bigger set H_2 we also have $A_{n+kr}(x, H_2)(\omega) > 2^k N(n)$. Further, since X_k is $\eta a^{2n+kr+1}$ dense, for any $x \in J_l$ we can find $x' \in X_1$ such that $|x - x'| < \eta a^{2n+kr} < \eta a^{n+kr}$. This proves (25).

It is remained to be proved that the event in (26) is measurable which is formally not straightforward since x is running over interval J_l.

First, we note that it is enough to prove that for any fixed $x' \in X_k$ the event

$$\left\{\forall x \in \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l : \; A_{n+kr}(x, H_2) \geq 2^k N(n)\right\}$$

is measurable since X_k is a finite set.

We have to take into consideration two facts. H_2 is a union of finite number of intervals and $A_{n+kr}(x, H_2)$ is a sum of finite number of indicator functions:

$$A_{n+kr}(x, H_2) = \sum_{Q_{\mathbf{i},\mathbf{j}}, |\mathbf{i}|=|\mathbf{j}|=n+kr} \mathbf{1}\left\{\Phi(Q_{\mathbf{i},\mathbf{j}}, x) \in H_2\right\}.$$

Therefore, the function $A_{n+kr}(\cdot, H_2)(\omega)$ for any ω is a jump function on T with finite number of jumps. Let $\{\iota_i : i \in \mathcal{I}\}$ denote the partition of T into the intervals on which $A_{n+kr}(\cdot, H_2)$ is constant. So, $A_{n+kr}(x, H_2)$ depends on the interval ι_i which x falls into. Therefore,

$$\left\{\forall x \in \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l : \; A_{n+kr}(x, H_2) \geq 2^k N(n)\right\} =$$
$$\left\{\forall i \in \mathcal{I} \text{ such that } \iota_i \cap \left[x' - \eta a^{n+kr}, x' + \eta a^{n+kr}\right] \cap J_l \neq \emptyset : \; A_{n+kr}(\iota_i, H_2) \geq 2^k N(n)\right\}.$$

The last event is a measurable function of finite number of random variables $\{J_l\} \cup \{\iota_i : i \in \mathcal{I}\} \cup \{Q_{\mathbf{i},\mathbf{j}}, |\mathbf{i}| = |\mathbf{j}| = n + kr\}$ hence measurable. $\square$

As a consequence of Fact 2 we can exchange X_1 with J_l and obtain

$$\mathbb{P}\left(A_{n+r}(x, H_2) > 2N(n), x \in J_l \mid \Omega_l\right) \geq 1 - (K^2)^{n+r} \tau^{N(n)} = 1 - a_1(n).$$

For $k = 0$, using Lemma 6, the definition of Ω_l and $H_1 \subset H_2$, we have

$$A_n(x, H_2)(\omega) \geq A_n(x, H_1)(\omega) \geq N(n) \quad \text{for all } \omega \in \Omega_l.$$

Therefore, we have (21) for $M = 1$:

$$\mathbb{P}\left(A_{n+kr}(x, H_2) > 2^k N(n), 0 \leq k \leq 1, x \in J_l \mid \Omega_l\right) \geq 1 - (K^2)^{n+r} \tau^{N(n)} =$$
$$1 - a_1(n) > (1 - a_1(n))(1 - a_0(n)).$$

Now, assume that we have proved (23) for $M - 1$. We will prove it for M. The simple fact that for every events A, B, C of positive probability we have: $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C) \cdot \mathbb{P}(B|C)$ yields

$$\mathbb{P}\left(A_{n+kr}(x, H_2) > 2^k \cdot N(n), \forall 0 \leq k \leq M, \; \forall x \in J_l \mid \Omega_l\right) =$$
$$\mathbb{P}\left(A_{n+Mr}(x, H_2) > 2^M \cdot N(n), \; \forall x \in J_l \mid A_{n+kr}(x, H_2) > 2^k \cdot N(n), \forall 0 \leq k \leq M - 1, \; \forall x \in J_l, \; \Omega_l\right) \cdot$$
$$\cdot \mathbb{P}\left(A_{n+kr}(x, H_2) > 2^k \cdot N(n), \forall 0 \leq k \leq M - 1, \; \forall x \in J_l \mid \Omega_l\right)$$

By induction, it is known that the second term on the right hand side is larger than $\prod_{k=0}^{M-1}(1-a_k(n))$. Now we use a similar argument, to the one applied as in the case $M=1$, for proving that the first term, on the right hand side in the displayed formula above, is larger than $1-a_M(n)$. Namely, let $\Omega_{l,M-1}$ denote the event in the condition of the first term:

$$\Omega_{l,M-1} := \left\{A_{n+kr}(x,H_2) > 2^k\cdot N(n), \forall 0 \leq k \leq M-1,\ \forall x \in J_l, \right\} \cap \Omega_l.$$

As in the proof of (22) we use Lemma 5. Let $n' = n+(M-1)r$, $C = A_{n+(M-1)r}(x,H_2)$ then for any $x \in J_l$ we have

$$\mathbb{P}\left(A_{n'+r}(x,H_1) \leq 2\cdot 2^{M-1}N(n) \mid \Omega_{l,M-1}\right) \leq \tau^{2^{M-1}N(n)}.$$

This can be proved in exactly the same way as (22) was proved. The continuation is also similar, we first take a dense set X_M and prove the counterpart of (24), that is,

$$\mathbb{P}\left(A_{n'+r}(x,H_1) > 2\cdot 2^{M-1}N(n), x \in X_M \mid \Omega_{l,M-1}\right) \geq 1-(K^2)^{n+Mr}\tau^{2^{M-1}N(n)} = 1-a_M(n).$$

Applying Fact 2 again yields

$$\mathbb{P}\left(A_{n+Mr}(x,H_2) > 2^M\cdot N(n),\ \forall x \in J_l \mid \Omega_{l,M-1}\right) \geq 1-a_M(n)$$

using $A_{n+Mr}(x,H_2) = A_{n'+r}(x,H_2)$. This finishes the proof of Lemma 7. $\square$

Now, we are ready to finish the proof of Theorem 2.

Proof of Theorem 2 . Using Lemma 7 we have

$$\mathbb{P}\left(\Pi(C_1-C_2) \text{ contains an interval } \mid \Omega_l\right) \geq \mathbb{P}\left(\Pi(C_1-C_2) \text{ contains } J_l \mid \Omega_l\right) \geq$$
$$\mathbb{P}\left(A_{n+Mr}(x,H_2) > 2^M\cdot N(n), \forall M,\ \forall x \in J_l \mid \Omega_l\right) > \prod_{k=0}^{\infty}(1-a_k(n)) > 1-2\sum_{k=0}^{\infty}a_k(n)$$

Getting rid of the condition, we obtain that

$$\mathbb{P}\left(\Pi(C_1-C_2) \text{ contains an interval }\right) =$$
$$\sum_{l=1}^{L}\mathbb{P}\left(\Pi(C_1-C_2) \text{ contains an interval } \mid \Omega_l\right)\mathbb{P}(\Omega_l) >$$
$$\left(1-2\sum_{k=0}^{\infty}a_k(n)\right)\sum_{l=1}^{L}\mathbb{P}(\Omega_l) = 1-2\sum_{k=0}^{\infty}a_k(n) > 1-\xi.$$

Since ξ can be chosen arbitrarily small this proves Theorem 2.

$\square$

5 Kernel of the branching process

In Sections 5 and 6, we prove the Main Lemma. In Section 5 we introduce several notations and determine support of the kernel of our branching process. The proof of Main Lemma starts in Section 6.

5.1 Types, stripes, and holes

Remember that we defined in Section 3.1 the renormalization operator $\Phi(Q, x)$. In the special case when Q_{ij} is a level-1 cylinder square which is the ith horizontally and the jth vertically we write $\Phi_{ij}(x) := \Phi(Q_{ij}, x)$. That is if the type of the ancestor is x then the type of the descendant is determined by Q_{ij} is denoted by $\Phi_{ij}(x)$, where

$$\Phi_{ij}(x) := \begin{cases} \dfrac{x + U_i^{(1)} - U_j^{(2)}}{a} & \text{if } e(x) \text{ intersects } Q_{ij}, \\ \Theta & \text{otherwise,} \end{cases} \tag{27}$$

where $\left(U_i^{(1)}, U_j^{(2)}\right)$ is the left bottom corner of Q_{ij}. Note that $U_i^{(1)}$ is independent of $U_j^{(2)}$.

Next, we are about to investigate the support of the kernel of our branching process m introduced in Section 3.4. Denote by $\phi_{ij}(x)$ the density function of $\Phi_{ij}(x)$. Note that it is not always a probability density function since it may have an atom at Θ. Recalling the argument in (11) we have

$$m(x, y) = \sum_{i,j=1}^{K} \phi_{ij}(x, y)\mathbf{1}\{(x, y) \in [-1, 1] \times [-1, 1]\}.$$

We deal with the support of m

$$\operatorname{supp} m = \bigcup_{i,j=1}^{K} \{(x, y) : -1 \le x, y \le 1,\ y \in \operatorname{supp} \phi_{ij}(x)\}.$$

We claim that $\operatorname{supp} m$ can be written in the following form

$$\operatorname{supp} m = \bigcup_{k=-K+1}^{K-1} S_k, \tag{28}$$

where

$$S_k = \bigcup_{s:1 \le s, s+k \le K} \{(x, y) : -1 \le x, y \le 1,\ y \in \operatorname{supp} \phi_{s+k,s}(x)\}, \tag{29}$$

is a slanted stripe, see Figure 5. Moreover, we will determine the boundaries of S_k the lines ℓ_{2k-1} and ℓ_{2k}, $k = -K, \dots, K$ for which

$$S_k = \{(x, y) : -1 \le x, y \le 1,\ \ell_{2k-1}(x) \le y \le \ell_{2k}(x)\} \tag{30}$$

Or first step toward this direction is determining the support of $\phi_{ij}(x)$ for any $i,j = 1,\dots K$. By construction, $\operatorname{supp} U_i \subset [a(i-1), 1-a(K-i)]$. More precisely, there exist $\alpha_i, \beta_i \geq 0$ such that

$$\min\{\operatorname{supp} U_i\} = a(i-1)+\alpha_i \quad \text{and} \quad \max\{\operatorname{supp} U_i\} = 1-a(K-i)-\beta_i.$$

Using (27), we have

$$\frac{1}{a}\left(x+\min U_i^{(1)}-\max U_j^{(2)}\right) \leq \frac{1}{a}\left(x+U_i^{(1)}-U_j^{(2)}\right) \leq \frac{1}{a}\left(x+\max U_i^{(1)}-\min U_j^{(2)}\right)$$

hence

$$\begin{aligned}\frac{1}{a}\left(x+a(i-1)+\alpha_i-1+a(K-j)+\beta_j\right) \leq \\ \frac{1}{a}\left(x+U_i^{(1)}-U_j^{(2)}\right) \leq \\ \frac{1}{a}\left(x+1-a(K-i)-\beta_i-a(j-1)-\alpha_j\right).\end{aligned}$$

As a conclusion we get

$$\begin{aligned}&\operatorname{supp}\phi_{ij} = \\ &\left[\frac{1}{a}\left(x+a(i-1)+\alpha_i-1+a(K-j)+\beta_j\right), \frac{1}{a}\left(x+1-a(K-i)-\beta_i-a(j-1)-\alpha_j\right)\right].\end{aligned} \tag{31}$$

Observe that if $i-j$ is constant, say $i=j+k$, then the support of $\Phi_{ij}(x)$ depends on k, see Figure 5, and the numbers $\alpha_i+\beta_j$ and $-\beta_i-\alpha_j$. This implies that we can arrange the support of m as it was stated in (28) and (29). Next, we determine the boundaries in (30) for which

$$S_k = \{(x,y) : -1 \leq x, y \leq 1,\ \ell_{2k-1}(x) \leq y \leq \ell_{2k}(x)\}.$$

Using the (31) and the remark after it one can find that the lines ℓ_{2k-1}, ℓ_{2k} can be written in the following form:

$$\begin{aligned}\ell_{2k-1}(x) &= \frac{1}{a}\left(x-1+a(k+K-1)+\min_{s:1\leq s+k, s\leq K}\{\alpha_{s+k}+\beta_s\}\right)\\ \ell_{2k}(x) &= \frac{1}{a}\left(x+1+a(k-K+1)-\min_{s:1\leq s+k, s\leq K}\{\beta_{s+k}+\alpha_s\}\right)\end{aligned}$$

An immediate calculation shows that

Lemma 8. *For every $x\in[-1,1]$ and for every $i,j=1,\dots,K$ if $\Phi_{ij}(x)\neq\Theta$ then*

$$(x,\Phi_{ij}(x)) \in S_{-K+1}\cup\dots\cup S_{K-1}.$$

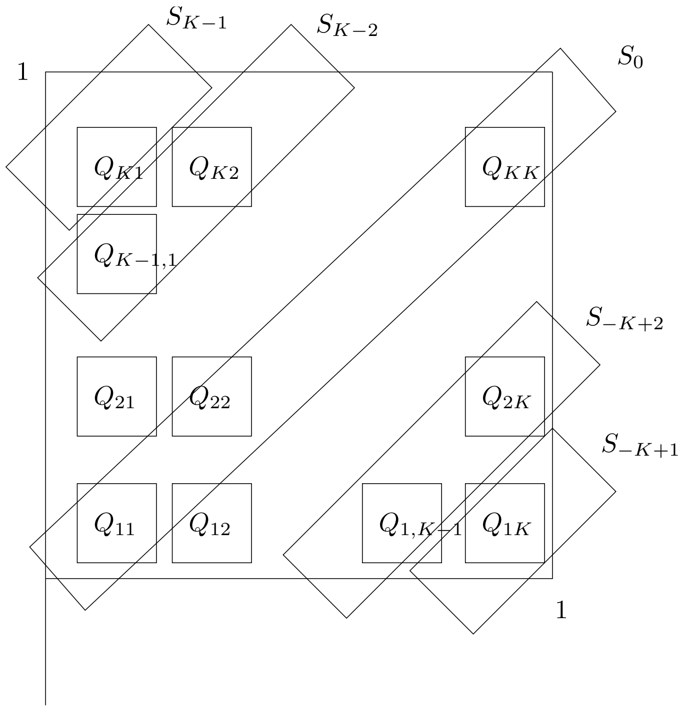

Figure 5: The level 1 cylinder squares and the connection between the cylinder squares and stripes $S_{-K+1},\dots,S_{K-1}$.

Let us call ℓ_j the graph of the function $\ell_j(x)$. For a point $(x_1,x_2)\in\mathbb{R}^2$ we write $\pi_m(x_1,x_2):=x_m$, $m=1,2$. Then we define $c^1,c^2>0$ by

$$-1+c^1:=\pi_1\left(\ell_{2(-K+1)}\cap\{y=x\}\right)$$

and

$$1-c^2=\pi_1\left(\ell_{2(K-1)-1}\cap\{y=x\}\right).$$

The functions $\ell_{2(-K+1)}(x),\ell_{2(K-1)-1}(x)$, these are the most left and the most right lines, have repelling fixed point $-1+c^1, 1-c^2$ respectively. Therefore

$$x\in\left[-1,-1+c^1\right)\cup\left(1-c^2,1\right]\Longrightarrow\exists n;\ e(x)\cap Q=\emptyset \text{ for all } Q\in\mathcal{S}_n. \tag{32}$$

Further,

$$x\in\left[-1,-1+c^1\right]\cup\left[1-c^2,1\right]\Longrightarrow\Phi_{ij}(x)\in\left[-1,-1+c^1\right]\cup\left[1-c^2,1\right]\ \text{ if } \Phi_{ij}(x)\neq\Theta. \tag{33}$$

5.2 The possible holes in the support of the kernel of $\mathcal{Z}$

We have seen in (32) that the branching process with ancestor type in the set $[-1,-1+c^1]$ or $[1-c^2,1]$ dies out in a finite number of generations almost surely. Therefore, it is reasonable to restrict the type space to $[-1+c^1+\varepsilon,1-c^2-\varepsilon]$ for some small positive ε. However, in some cases we have to make further restrictions. Namely, for $i = -K+2, \dots, K-1$ we define

$$u^i := \pi_1 \left(\ell_{2i-3} \cap \left\{ y = 1 - c^2 \right\} \right), v^i := \pi_1 \left(\ell_{2i} \cap \left\{ y = -1 + c^1 \right\} \right), \tag{34}$$

see Figure 6. If $u^i < v^i$ for some $i = -K+2, \dots, K-1$ holds then for $x \in [u^i, v^i]$ the set

$$E_1(x) := \{ y : m(x,y) > 0 \} \tag{35}$$

is contained in $[-1, -1+c^1] \cup [1-c^2, 1]$, for example u^0 and v^0 have this property in Figure 6. This and (32) imply that the process dies out in finitely many steps for $x \in (u^i, v^i)$ (see Figure 6). Therefore, we have to make more restrictions on our type space $[-1+c^1+\varepsilon, 1-c^2-\varepsilon]$, for example the interval $[u^i, v^i]$ has to be thrown out. These further restrictions will be determined precisely in the next section.

6 Construction of a uniformly positive kernel

In this section, we will prove the Main Lemma. Now, we state it again in a more informal way.

Theorem 4 (**Main Lemma**). *Let us be given a homogeneous RIFS $\mathcal{G}$, and denote by m the kernel determined by $\mathcal{G}$ in Section 5. There exist a set of open intervals, $T(0)$ and a real number $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the set*

$$T(\varepsilon) := T(0) \setminus B(\partial T(0), \varepsilon)$$

is a type space such that

1. *$T(\varepsilon)$ is consists of as many intervals as $T(0)$ does.*
2. *condition (C1) satisfies, that is, the kernel*

$$m^\varepsilon := m \,|_{T(\varepsilon) \times T(\varepsilon)}$$

   is uniformly positive and bounded.

3. *the Perron-Frobenius eigenvalue of $m \,|_{T(\varepsilon)\times T(\varepsilon)}$ is larger than 1.*
4. *the corresponding eigenfunction is continuous on $T(\varepsilon)$.*

The construction of type space $T(\varepsilon)$ ($\varepsilon \in (0, \varepsilon_0)$) consists of two steps. We will call any open subset of $[-1, 1]$ a pre-type space. First we inductively construct a sequence of pre-type spaces $T^0 \supset T^1 \supset \cdots \supset T^l$. Those elements of T^l which are "far" from the endpoints of the components of T^l satisfy (36). (We define $T(0) := T_l$.) Unfortunately, the same does not hold for the points close the the boundary of the components of T^l. So, as a second step of the construction of T we remove a small neighborhood of the boundary of T^l from T^l. So we obtain $T(\varepsilon)$.

Lemma 9. *There exist a restriction of the pre-type space $(-1+c^1, 1-c^2)$, $T(0)$ and a real number $\varepsilon_1 > 0$ such that the kernel m^ε of the branching process $\mathcal{Z}$ with type space $T(\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_1)$ satisfies*

$$\forall \varepsilon \in (0,\varepsilon_1) \exists \kappa(\varepsilon) > 0 \text{ such that } \forall x \in T \text{ the set } E_1(x) = \{y : m^\varepsilon(x,y) > 0\} \text{ contains an interval of length at least } \kappa(\varepsilon). \quad (36)$$

Further, for any $\varepsilon \in (0, \varepsilon_0)$ $T(\varepsilon)$ consists of as many interval as $T(0)$.

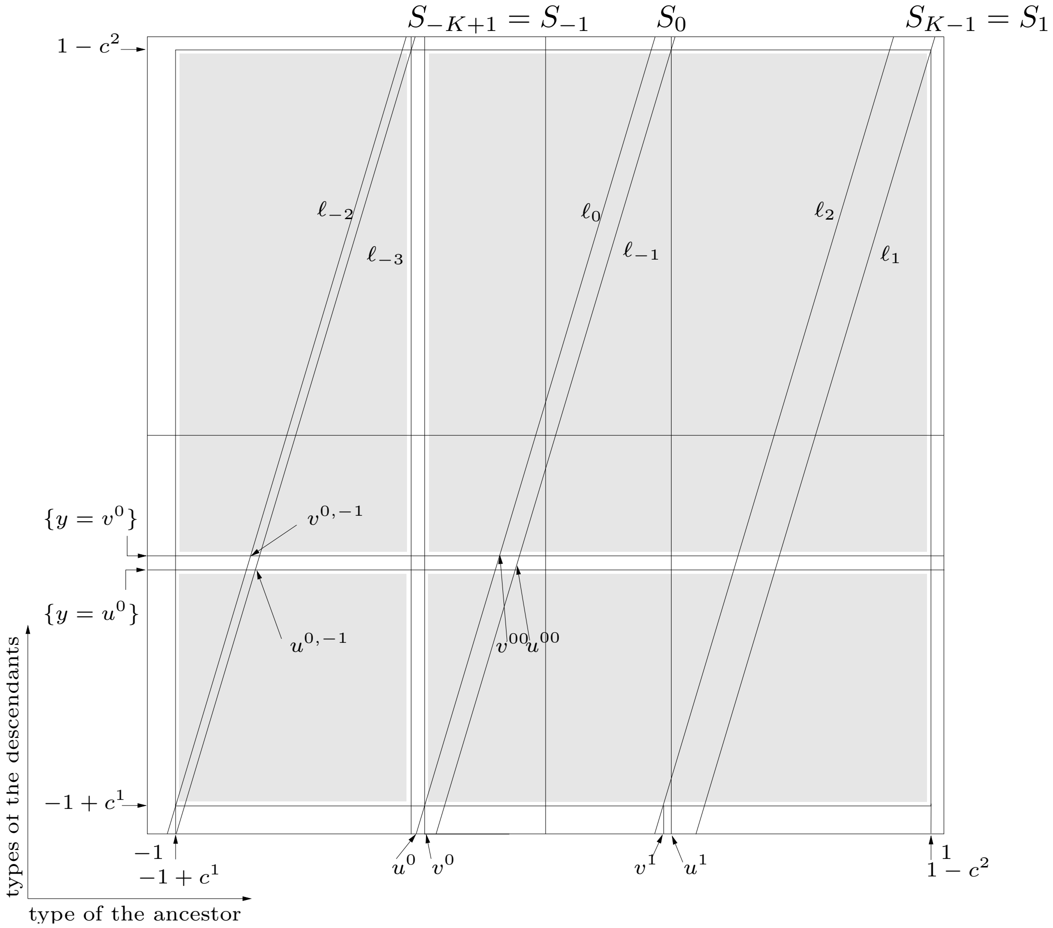

Figure 6: Some points and lines related to the kernel $m(x, y)$ if $l = 1$

Proof of Lemma 9. We recall that u^k, v^k, $-K+2 \le k \le K-1$ were defined in (34) and we take the pre-type space $T^0 := (-1+c^1, 1-c^2)$. If $v^k < u^k$ for any $-K+2 \le k \le K-1$ then we define $l := 0$.

If for some $-K+2 \le k \le K-1$ $u^k \le v^k$, then we have to subtract $[u^k, v^k]$ from the pre-type space T^0 to insure that (36) holds (cf. Figure 6). Let $\mathcal{K}_1$ be the set of such i's, that is,

$$\mathcal{K}_1 = \{k : \ -K+2 \le k \le K-1, \ u^k \le v^k\}.$$

Remark that if $x \in [u^k, v^k]$ ($k \in \mathcal{K}_1$) then the types of descendants of x fall into the set $[-1,1] \setminus T^0$. So, we restrict ourselves to the next level pre-type space:

$$T^1 = T^0 \setminus \left(\bigcup_{i \in \mathcal{K}_1} [u^i, v^i] \right).$$

We define the second generation endpoints u^{ik} and v^{ik} as follows

$$u^{ik} = \pi_1(\{y = u^i\} \cap \ell_{2k-1}) \text{ and } v^{ik} = \pi_1(\{y = v^i\} \cap \ell_{2k})$$

for any $i \in \mathcal{K}_1$ and $-K+1 \le k \le K-1$, see Figure 6. If $v^{ik} < u^{ik}$ for any $i \in \mathcal{K}_1$ and $-K+1 \le k \le K-1$, then we define $l := 1$. Otherwise, if $u^{ik} \le v^{ik}$ for some $-K+2 \le i \le K-1$ and $-K+1 \le k \le K-1$ we have to subtract $[u^{ik}, v^{ik}]$ from the pre-type space T^1 because if $x \in [u^{ik}, v^{ik}]$ then the types of descendants of x fall into the set $[-1,1] \setminus T^1$, that is, into an already removed set. So, let $\mathcal{K}_2$ be a subset of $\mathcal{K}_1 \times \{-K+1, \dots, K-1\}$ such that if $\mathbf{i} \in \mathcal{K}_2$ then $u^{\mathbf{i}} \le v^{\mathbf{i}}$. In this way, we define the next level pre-type space:

$$T^2 = T^1 \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{K}_2} [u^{\mathbf{i}}, v^{\mathbf{i}}] \right).$$

We continue defining the sets T^r and the endpoints of the subtracted intervals as follows: for $\mathbf{i} \in \mathcal{K}_{r-1}$ and $-K+1 \le k \le K-1$ let

$$\begin{aligned} u^{\mathbf{i}k} &= \pi_1(\{y = u^{\mathbf{i}}\} \cap \ell_{2k-1}), \quad \text{and} \\ v^{\mathbf{i}k} &= \pi_1(\{y = v^{\mathbf{i}}\} \cap \ell_{2k}). \end{aligned} \tag{37}$$

Let

$$\mathcal{K}_r = \{\mathbf{i}k \in \mathcal{K}_{r-1} \times \{-K+1, \dots, K-1\} : \; u^{\mathbf{i}k} \le v^{\mathbf{i}k}\}.$$

If $\mathcal{K}_r = \emptyset$, then $l = r-1$. If $\mathcal{K}_r \ne \emptyset$, then we remove the intervals $[u^{\mathbf{i}k}, v^{\mathbf{i}k}]$ for $\mathbf{i}k \in \mathcal{K}_r$ from the type space. Thus, let

$$T^r = T^{r-1} \setminus \left(\bigcup_{\mathbf{i} \in \mathcal{K}_r} [u^{\mathbf{i}}, v^{\mathbf{i}}] \right). \tag{38}$$

This construction ends at step l for some l because there exists an l such that

$$\text{for any } \mathbf{i}k \in \mathcal{K}_l \times \{-K+1, \dots, K-1\} \; v^{\mathbf{i}k} < u^{\mathbf{i}k}. \tag{39}$$

Now, we will prove that there exists an l with this property. For $\mathbf{i} \in \mathcal{K}_r$ for some r let

$$\varrho^{\mathbf{i}} = v^{\mathbf{i}} - u^{\mathbf{i}}$$

the length of the removed interval $[u^{\mathbf{i}}, v^{\mathbf{i}}]$. By construction, see also the left hand side of Figure 8, one can easily check that for $\mathbf{i} \in \mathcal{K}_r$ (for some r) and $k \in \{-K+1, \dots, K-1\}$

$$\varrho^{\mathbf{i}k} = a(\varrho^{\mathbf{i}} - (\ell_{2k}(x) - \ell_{2k-1}(x))). \tag{40}$$

It is enough to prove that after finite number of steps the left hand side becomes negative. For showing this we recursively define

$$\rho_{r+1} = a(\rho_r - \min_{k\in\{-K+1,\ldots,K-1\}} (\ell_{2k}(x) - \ell_{2k-1}(x))), \quad \rho_1 = \max_{k\in\{-K+2,\ldots,K-1\}} (v^k - u^k).$$

By construction $\varrho^{\mathbf{i}} \leq \rho_r$ for any $\mathbf{i} \in \mathcal{K}_r$. Since there exists an r for which $\rho_{r+1} < 0$ it is clear that there exist an l for which (39) holds.

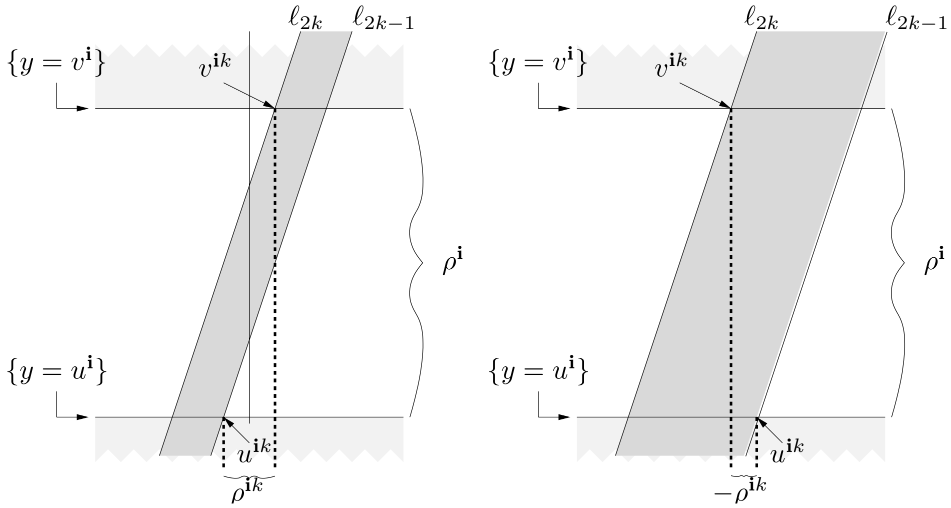

Figure 7: The computation of $\varrho^{\mathbf{i}k}$

We can represent T^l as follows

$$T^l = \bigcup_{i=1}^{\tau} (\alpha_i, \beta_i).$$

for some positive integer τ.

We need further restrictions because around the endpoints α_i, β_i condition (36) is not satisfied. Therefore we remove sufficiently small intervals from both ends of each intervals of T^l. Namely, we define the type space of the process by

$$T(\varepsilon) := \bigcup_{i=1}^{\tau} [\alpha_i + \varepsilon, \beta_i - \varepsilon], \tag{41}$$

where

$$0 < \varepsilon < \varepsilon_1, \tag{42}$$

where the value ε_1 will turn out during the proof below. The first bound for ε_1 is that $T(\varepsilon)$ has to consists of as many intervals as $T(0)$ so by introducing

$$\varepsilon^{(1)} = \frac{1}{2} \min_{i=1,\ldots,r} \{\beta_i - \alpha_i\}$$

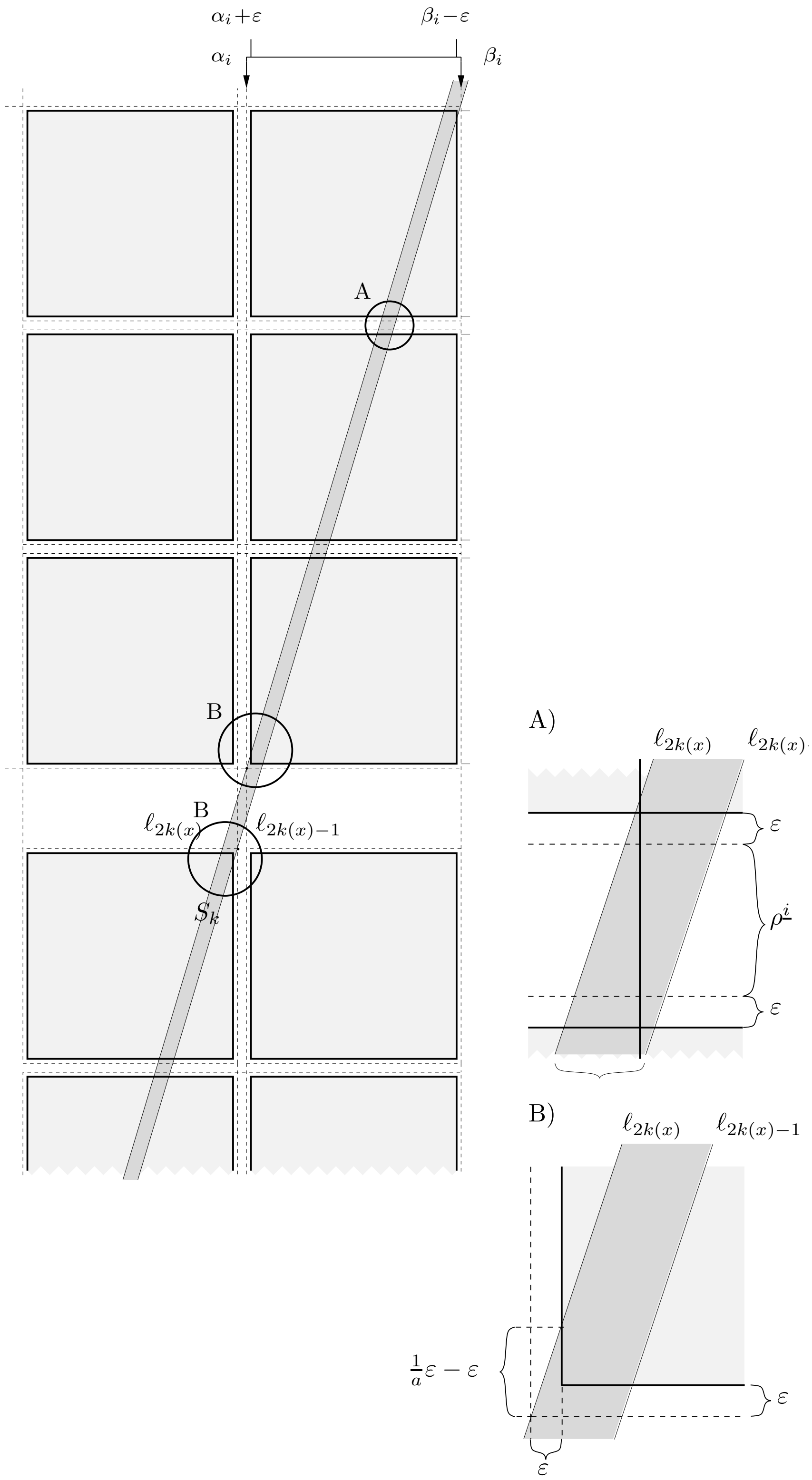

Figure 8: Stripe S_k and level l squares.

the following inequality has to hold

$$\varepsilon_1 < \varepsilon^{(1)}.$$

Now, we prove that (36) holds and find further bounds for ε_1. That is, we want to estimate the length of the longest interval in $E_1(x)$ from below. The argument uses only elementary geometry.

For any $x \in T(\varepsilon)$ there is an index $k(x) \in \{-K+1, \dots, K-1\}$ such that the intersection $E_1^*(x) = E_1(x) \cap (\ell_{2k(x)-1}(x), \ell_{2k(x)}(x))$ is not empty. If there are more indices pick the one for which the measure of the intersection is the largest. Note that the vertical line through x intersects the stripe $S_{k(x)}$ in a (vertical) interval of length $\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x)$.

Since there are many holes in $T(\varepsilon)$, for some $x \in T(\varepsilon)$, the set $E_1(x)$ may consists of several subintervals. We prove that the maximum length of these intervals is uniformly bounded away from zero.

Fix an arbitrary $x \in T(\varepsilon)$. We separate four cases.

(1) Assume that $(\ell_{2k(x)-1}(x), \ell_{2k(x)}(x))$ is contained in $[\alpha_j + \varepsilon, \beta_j - \varepsilon]$ for some j. By elementary geometry we obtain that $E_1(x)$ is an interval of length $\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x)$. This case does not affect the choice of ε_1. Hence, $E_1(x)$ is an interval of length at least

$$\kappa^{(1)}(\varepsilon) := \min_{-K+1 \leq k \leq K-1} \{\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x)\}.$$

(2) Assume that (1) does not hold. Assume that $E_1(x) = (\ell_{2k(x)-1}(x), \ell_{2k(x)}(x)) \cap T(\varepsilon) \supset [\alpha_j + \varepsilon, \beta_j - \varepsilon]$ for some j. In this case $E_1(x)$ contains an interval of length $\beta_j - \alpha_j - 2\varepsilon$. If

$$\varepsilon < \varepsilon^{(2)} := \frac{1}{2} \min_{j \in \{1,\dots,\tau\}} \{\beta_j - \alpha_j\},$$

then we can say that $E_1(x)$ contains an interval of length at least $\kappa^{(2)}(\varepsilon) := 2\varepsilon^{(2)} - 2\varepsilon$.

(3) Assume that neither (1) nor (2) holds. Assume that for some $\mathbf{i} \in \mathcal{K} := \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_l$

$$(\ell_{2k(x)-1}(x), \ell_{2k(x)}(x)) \cap [u^{\mathbf{i}}, v^{\mathbf{i}}] \neq \emptyset \quad \text{but} \quad u^{\mathbf{i}k(x)} > v^{\mathbf{i}k(x)}.$$

Therefore, $[v^{\mathbf{i}k(x)}, u^{\mathbf{i}k(x)}] \subset T$. This is the case that is depicted in Figure 8A and the right sub figure in Figure 7. Hence

$$u^{\mathbf{i}k(x)} - v^{\mathbf{i}k(x)} = -a(\varrho^{\mathbf{i}} - \min_{k \in \{-K+1,\dots,K-1\}} (\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x))) > 0.$$

Elementary geometry shows that $E_1(x)$ contains an interval of length $-\frac{1}{2}\frac{1}{a}a(\varrho^{\mathbf{i}} - (\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x)))$. If ε is selected so that

$$\varepsilon < \varepsilon^{(3)} := \frac{1}{2a} \min_{i \in \mathcal{K}, k} \left\{-a(\varrho^{\mathbf{i}} - (\ell_{2k}(x) - \ell_{2k-1}(x)))\right\},$$

then we can say that $E_1(x)$ contains an interval of length at least

$$\kappa^{(3)}(\varepsilon) = 2\varepsilon^{(3)} - 2\varepsilon.$$

(4) Assume that nor (1) neither (2) hold. Assume that for some $\mathbf{i} \in \mathcal{K} := \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_l$

$$(\ell_{2k(x)-1}(x), \ell_{2k(x)}(x)) \cap [u^{\mathbf{i}}, v^{\mathbf{i}}] \neq \emptyset \quad \text{and} \quad u^{\mathbf{i}k(x)} \leq v^{\mathbf{i}k(x)},$$

that is, $\mathbf{i}k \in \mathcal{K}$ as well. This is the case that is depicted in Figure 8B. This means that $(\ell_{2k(x)-1}(x), \ell_{2k(x)}(x))$ contains an interval of length

$$\kappa^{(4)}(\varepsilon) := \min_{j \in \{1,\ldots,\tau\}} \{\beta_j - \alpha_j\} \wedge \left(\frac{1}{a}\varepsilon - \varepsilon\right) > \min_{j \in \{1,\ldots,\tau\}} \{\beta_j - \alpha_j\} \wedge \varepsilon(K-1)$$

if $\varepsilon > 0$. Hence, case (4) does not restrict the set of possible ε.

Summarizing these cases, if

$$\varepsilon_1 := \min\{\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}\}$$

$0 < \varepsilon < \varepsilon_1$, then (36) holds with

$$\kappa(\varepsilon) = \min\{\kappa^{(1)}(\varepsilon), \kappa^{(2)}(\varepsilon), \kappa^{(3)}(\varepsilon), \kappa^{(4)}(\varepsilon)\}.$$

□

We have to remark that by the construction of the pre-type spaces $T_0, T_1, \ldots, T_l$ in Lemma 9 we obtain the following (recall that $T(0) = T_l$)

Fact 3. *If $x \in [-1,1] \setminus T(0)$ then the types of any offspring of x fall into $[-1,1] \setminus T(0)$ with probability 1.*

We will now deal with the problem of still having a kernel with largest eigenvalue larger than 1.

Lemma 10. *Let m^ε be the kernel in Lemma 9 with type space $T(\varepsilon)$, $\varepsilon \in (0, \varepsilon_1)$. There exists an ε_0, $0 < \varepsilon_0 \leq \varepsilon_1$ such that if $\varepsilon \in (0, \varepsilon_0)$, then the largest eigenvalue of m^ε is larger than 1.*

Proof. It is enough to prove that $K^2 a$ is an eigenvalue of the operator $\mathcal{T}_l$ with eigenfunction $h(x) = \mathbf{1}_{T^l}(x)$ where T^l is defined in the proof of Lemma 9:

$$\begin{aligned}
\mathcal{T}_l h(y) &= \int_{\mathbb{R}} h(x) m(x,y)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} h(x) \Big(\sum_{i,j=1}^{K} \phi_{ij}(x,y) \Big) \mathbf{1}_{T^l}(y)\,\mathrm{d}x \\
&= K^2 a h(y) \int_{T^l} \sum_{i,j=1}^{K} \phi_{ij}(ay - x)\,\mathrm{d}x \\
&= K^2 a h(y),
\end{aligned}$$

provided we show that for all $i, j = 1, \ldots, K$ and for all $y \in T^l$

$$\int_{T^l} \phi_{ij}(ay - x)\,\mathrm{d}x = 1.$$

So we have to show that for all $y \in T^l$ and for $i, j = 1, \ldots, K$

$$\{x : \phi_{ij}(ay - x) > 0\} \subset T^l. \tag{43}$$

This holds since we constructed the intermediate type space T^l so that this property is satisfied, see the left figure in Figure 7: we have subtracted intervals of the form $[u^{\mathbf{i}k}, v^{\mathbf{i}k}]$ in (38) during the construction of successive intermediate type spaces T^r, $r = 1, \ldots, l$. If $y \in T^r$ then each interval of the form of $[u^{\mathbf{i}k}, v^{\mathbf{i}k}]$ is disjoint of $(\ell_{2k}^{-1}(y), \ell_{2k-1}^{-1}(y))$ for all $y \in T^l$ and $k \in \{-J+1, \ldots, K-1\}$. Therefore, for any $y \in T^l$ we have $(\ell_{2k}^{-1}(y), \ell_{2k-1}^{-1}(y)) \subset T^l$. Further, for any $i, j = 1, \ldots, K$ there exists a positive integer (and corresponding stripe) $k_{ij} \in \{-K+1, \ldots, K-1\}$ such that

$$\{x : \phi_{ij}(ay - x) > 0\} = (\ell_{2k_{ij}}^{-1}(y), \ell_{2k_{ij}-1}^{-1}(y)).$$

Hence, (43) holds.

The conclusion of the lemma follows from a simple fact noted by Larsson [6]: If the two kernels m^0 and m^ε are close to each other in L^2 sense, then the eigenvalues of the operators $\mathcal{T}_0$ and $\mathcal{T}_\varepsilon$, determined by the kernels m^0 and m^ε respectively, are close to each other.

Finally, let $\varepsilon_0 \le \varepsilon_1$ be chosen such that for any $\varepsilon \in (0, \varepsilon_0)$ the eigenvalue of $\mathcal{T}_\varepsilon$ is larger than 1.

□

Lemma 11. *Let $T(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$ be as in Lemma 10. Then there exists an n such that for all $x \in T(\varepsilon)$, $\{y : m_n^\varepsilon(x, y) > 0\} = T(\varepsilon)$.*

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$ and for simplicity let $\kappa := \kappa(\varepsilon)$, $m := m^\varepsilon$, and $T := T(\varepsilon)$.

We will prove the lemma in two steps. We define

$$E_n(x) := \{y : m_n(x, y) > 0\}.$$

Step 1 $\forall x \in T, \exists i, n$ such that $[\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_n(x)$ implies that $E_{n+l}(x) = T$.

Step 2 There exists an N such that for every $x \in T$ we can find a positive integer $n(x) \le N$ such that the following holds

$$\exists i,\ [\alpha_i + \varepsilon, \beta_i - \varepsilon] \subset E_{n(x)}(x).$$

As a corollary of these two statements we obtain the assertion of the lemma holds with the choice of $n = N + l$. Namely, for any $x \in T$ we have $E_{N+l}(x) = T$.

Next, we prove Step 1 and Step 2 separately. The ideas of th proof are borrowed from [2].

Proof of Step 1: To verify *Step 1* first we observe that we can derive $E_{n+1}(x)$ from $E_n(x)$ by means of the equation

$$m_{n+1}(x,y) = \int_T m_n(x,z) m_1(z,y)\,\mathrm{d}z,$$

which implies

$$E_{n+1}(x) = \bigcup_{z\in E_n(x)} E_1(z).$$

Using this formula and (28), (29), and (30), we obtain that

$$E_{n+1}(x) = \bigcup_{z\in E_n(x)} E_1(z) = \bigcup_{z\in E_n(x)} \left(\bigcup_{k=-K+1}^{K-1} (\ell_{2k-1}(z), \ell_{2k}(z)) \right) \cap T. \tag{44}$$

For some $i = 1, \ldots, \tau$ fix $[\alpha_i + \varepsilon, \beta_i - \varepsilon]$. First, we define $\alpha_{i,l-r}$ and $\beta_{i,l-r}$ for $r = 0, \ldots, l$ inductively. For $r = 0$ let $(\alpha_{i,l}, \beta_{i,l}) := (\alpha_i, \beta_i)$. Assume that we have already defined $(\alpha_{i,l-r}, \beta_{i,l-r})$. Using (37) we define $\alpha_{i,l-(r+1)}$ and $\beta_{i,l-(r+1)}$ as the unique numbers satisfying:

$$\begin{aligned} \alpha_{i,l-r} &= \pi_1\big(\big\{(x,y)\colon y = \alpha_{i,l-(r+1)}\big\} \cap \ell_{2k(r)-1}\big)\,, \\ \beta_{i,l-r} &= \pi_1\big(\big\{(x,y)\colon y = \beta_{i,l-(r+1)}\big\} \cap \ell_{2k(r)}\big)\,, \end{aligned} \tag{45}$$

where $k(r) \in \{-K+1, \ldots, K-1\}$. Then by the construction we have $(\alpha_{i,0}, \beta_{i,0}) = (-1+c^1, 1-c^2)$. Note that it may happen that $(\alpha_{i,l-s}, \beta_{i,l-s}) = (-1+c^1, 1-c^2)$ for $s > 1$. Let $x \in T$. According to the assumption of *Step 1* we can find i, n such that

$$[\alpha_i + \varepsilon, \beta_i - \varepsilon] = (\alpha_i, \beta_i) \cap T \subset E_n(x) \tag{46}$$

holds. Using induction we prove that

$$E_{n+r}(x) \supset (\alpha_{i,l-r}, \beta_{i,l-r}) \cap T \text{ for } 0 \le r \le l. \tag{47}$$

Namely, for $r = 0$ the assertion in the induction is identical to (46). Now we suppose that (47) holds for $r < l$. By (44) and (45) we have

$$\begin{aligned} E_{n+r+1}(x) &= \bigcup_{z\in E_{n+r}(x)} E_1(z) \\ &\supset \bigcup_{z\in(\alpha_{i,l-r},\beta_{i,l-r})\cap T} (\ell_{2k(r)-1}(z), \ell_{2k(r)}(z)) \cap T \\ &= (\alpha_{i,l-(r+1)}, \beta_{i,l-(r+1)}) \cap T, \end{aligned}$$

which completes the proof of (47). We apply (47) for $r = l$. This yields that $E_{n+l} = (-1+c^1, 1-c^2) \cap T = T$ holds.

Proof of Step 2: First, in Case (2) in the end of the proof of Lemma 9 $E_1(x) = [\alpha_i + \varepsilon, \beta_i - \varepsilon]$ for some i. So, in this case the statement of the lemma is settled.

Second, observe that the largest interval in $E_1(x)$ either has an endpoint that is an endpoint of a connected component of T (this happens in case (3) and (4) in the end of the proof of Lemma 9) or $E_1(x) = (\ell_{2k(x)-1}(x), \ell_{2k(x)}(x))$ (which is case (2) in the same proof). However, in the last case using (44), after N_1 steps, where N_1 is the smallest solution of the inequality

$$\left(\frac{2}{a}\right)^{N_1} \cdot \min_{k\in\{-K+1,\dots,K-1\}} (\ell_{2k(x)}(x) - \ell_{2k(x)-1}(x)) > \max_{i=1,\dots,\tau} (\beta_i - \alpha_i - 2\varepsilon)$$

we obtain that the largest interval contained in $E_{N_1}(x)$ has an endpoint of a connected component of T (see Figure 8) and its length is longer than κ. In this way because of the symmetry between the endpoints of the connected components of T from now on we may assume that $[\alpha_i + \varepsilon, \alpha_i + \varepsilon + w_1) \subset E_1(x)$ where $w_1 \geq \kappa$. Using (44) we can write

$$\begin{aligned} E_2(x) &\supset \bigcup_{z\in[\alpha_i+\varepsilon,\alpha_i+\varepsilon+w_1)} (\ell_{2k_1-1}(z), \ell_{2k_1}(z)) \cap T \\ &= (\ell_{2k_1-1}(\alpha_i + \varepsilon), \ell_{2k_1}(\alpha_i + \varepsilon + w_1)) \cap T \\ &= [\alpha^{(2)} + \varepsilon, \alpha^{(2)} + \varepsilon + w_2) \cap T \end{aligned} \tag{48}$$

for some $k_1 \in \{-K+1, \dots, K-1\}$, left endpoint $\alpha^{(2)} \in T$ and $w_2 > \frac{1}{a}w_1 \geq \frac{1}{a}\kappa$. If $[\alpha^{(2)} + \varepsilon, \alpha^{(2)} + \varepsilon + w_2) \cap T$ does not contain a connected components of T then we can inductively define for $E_n(x)$, $n \geq 3$ we can define inductively k_n, left endpoint $\alpha^{(n)}$, and length w_n in the same way as above. Observe that $w_n > \left(\frac{1}{a}\right)^{n-1}\kappa$ for any $n \geq 2$. Let N_2 the smallest solution of the inequality

$$\left(\frac{1}{a}\right)^{N_2-1} \kappa > \max_{i=1,\dots,\tau} (\beta_i - \alpha_i - 2\varepsilon).$$

Then $E_{N_2}(x)$ surely contains a connected component of T. However, it may happen that for $n < N_2$ $E_n(x)$ contains a connected components of T.

Let $N = N_1 + N_2$. Then $E_N(x)$ contains a connected component of T. □

Proof of Main Lemma. Let ε_0 be the same as in Lemma 10 and let $T(\varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$ the type space constructed in Lemma 9. Then we have the followings:

Statement (1) has been proved in Lemma 9.

Statement (2) follows from Lemma 11. Indeed, the uniformly positivity has been proved there, the boundedness of m^ε follows from the fact that m^ε is continuous on the compact set $T(\varepsilon)$.

Statement (3) has been proved in Lemma 10.

Statement (4) follows from the fact that m^ε is continuous on $T(\varepsilon) \times T(\varepsilon)$. Indeed, denoting by f the eigenfunction, we have $\varrho f(x) = \int m^\varepsilon(x, y) f(y)\, \mathrm{d}y$ and m^ε continuous in x. □

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