# Inaccuracy of Availability Metrics Estimated by the Serial-Parallel Model

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***Abstract*—Provisioning QoS network connections with the desired availability relies on estimating the availability of the connections in advance. The redundancy of the protection – which denotes how many failures can be survived – gives a good hint on the availability, thus lower bound estimation can be carried out. If we want to achieve a more accurate approximation, we have to use other methods. The Serial-Parallel availability modeling and calculation method, based on the availability metrics of the components of the connection, offers a fast estimation, with the complexity of $O(n)$ in case of $n$ components. However, the result of this estimation can be inaccurate since the model does not take into account the overlapping of components, i.e., when a component is member of more different series.**

**In this paper we analyze the inaccuracy of the Serial-Parallel method. We prove that the estimated availability is always less than the exact one, define an upper bound onto the inaccuracy of the estimated unavailability and show where does this inaccuracy converge by increasing the availability of the network components.**

***Index Terms*—availability, conditional probability, protection.**

## I. Introduction

Measuring and estimating the availability of network devices and connections is important since the provisioned network connections have to fulfill predefined availability requirements. The availability of a network device can be improved by extending its *mean time to failure* (MTTF) or by shortening its *mean time to repair* (MTTR) attribute. Assuming these attributes are predefined, generally, high connection availability can be achieved by setting up redundant backup paths along the working path, which are protecting parts (links or segments) of it or the whole path (end-to-end protection). However, the more complex the protection is, the more resources the connection requires [1], [2], and the more difficult is to derive its availability.

The availability of connection using dedicated protection can be estimated accurately, since the backup paths do not interfere. However, if the backup resources are *shared*, the availability estimation may become a complex problem. Exhaustive availability evaluation has to enumerate each possible network failure state variation, which can be carried out only for small systems. With stratified sampling we can eliminate the state space while achieving still a good approximation [4]. In [5], [6], [7] routing algorithms for shared protection with guaranteed availability are proposed without evaluating the exact connection availability.

The $p$-cycle protection scheme [9] provides a special way resource sharing, and in [8] we already have presented a fast evaluation method of the accurate availability for $p$-cycle-protected connections. Among the numerical results of [8] we compared the accurate availability to the availability estimated with the well-known Serial-Parallel (*S-P*) method[1] [10] and the experienced behaviour of the inaccuracy lead us to study the *S-P* heuristic more deeply in link-protected connections.

The rest of the paper is organized as follows. In the next section we define the scope of the work and introduce the notation for our availability model. In Sect. III we examine *S-P* inaccuracy with single link overlaps, next, in Sect. IV we extend these single overlaps to multiple link overlaps and deduce lower, upper bounds and limit value of the error of *S-P* method. Illustrative examples are studied in Sect. V, and finally, Sect. VI summarizes the results of the work.

## II. Notation and Scope of the Work

The network consists of atomic components which may fail. The whole set of these components is denoted by $E$, while for denoting a single component we use $e$ ($e \in E$) which is used usually with an index or other special markings (e.g., $e_i$, $e^*$, etc.) In our availability model we assume that each network component $e$ has two states ($S(e)$): it may be either operational (up, $S(e) = 1$) or in failure (down, $S(e) = 0$) state, and the state of each component is independent from the state of any other component ($\forall e', e'' \in E, e' \neq e'' : S(e')|_{S(e'')} = S(e')$). The availability of a component is a

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[1]The principle behind the Serial-Parallel method is that if dual-state (up/down) components are connected serially, the series is up only if *all* the components are up; whereas a block of parallelly switched components requires only a *single* component to be in up state.

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probability metric indicating that the component is in up state: $A(e) = P(S(e) = 1)$, whereas the unavailability is the inverse of the availability: $U(e) = P(S(e) = 0) = 1 - A(e)$.

We examine individual connections. The connection is denoted by $conn$, using the set components $C \subset E$. The state and the availability metric is also defined for connections. We assume that the state of the connection depends only on the atomic states of its components. In other words, the state of a foreign component does not influence the state of the connection: $\forall e \notin C : S(conn)|_{S(e)=1} = S(conn)|_{S(e)=0}$. Note, that this assumption constrains the scope of the work, since the most of the protection strategies that share resources, do not fulfill this requirement. Although the *p*-cycle protection also shares resources, it is a kind of *self-sharing*, i.e., the shared resources themselves form the backup paths, thus they all – as shared components, which may influence the state of the connection – are member of set $C$ [8]. This is an important remark since the results of this work were applied first of all onto *p*-cycle protected networks.

The state of the connection – derived merely from the state of the components – is also binary: the connection is either up ($S(conn) = 1$) or down ($S(conn) = 0$). This means that we do not take into account reconfiguration transients triggered by failures on the operational (working) path, when the connection, or a segment of it (in case of segment-protections [11]), has to be switched to an alternate (backup) path. For the availability of the connection and its approximation we use the following notations: $A_{ACC} = P(S(conn) = 1)$ is the *accurate* availability estimation with the corresponding $U_{ACC} = 1 - A_{ACC}$ unavailability, whereas $A_{SP}$ (with $U_{SP} = 1 - A_{SP}$) is the availability metric of the connection estimated by the Serial-Parallel method.

We assume that the connection is a series of protected links. The length of the connection is $n$, meaning that along the default path $n+1$ nodes ($0^{\text{th}}$ is the source node, $n^{\text{th}}$ is the destination) are connected with $n$ links. If we split the connection at the $i^{\text{th}}$ node into two, we get $conn_i^H$ and $conn_i^T$ as the *head* and the *tail* of the connection, with components enumerated in set $H_i = \{e_{h1}, e_{h2}, ...e_{hh}\}$ and set $T_i = \{e_{t1}, e_{t2}, ...e_{tt}\}$, so that the concatenating the tail to the head results in the original connection without any loss or surplus: $\forall 0 \le i \le n : conn = conn_i^H.conn_i^T$. Moreover the head and the tail part inherit the protection of the member links, resulting in $S(conn) = 1 \iff S(conn_i^H) = 1 \wedge S(conn_i^T) = 1$. Still $A_{ACC}(conn) \neq A_{ACC}(conn_i^H) \cdot A_{ACC}(conn_i^T)$, except for cases when $S(conn_i^H)$ and $S(conn_i^T)$ are *independent*. As we assumed that the states of the atomic components are independent, dependency between $S(conn_i^H)$ and $S(conn_i^T)$ can be observed if and only if $conn_i^H$ and $conn_i^T$ have some components in common. We call them *overlapping* links (defined by the intersection $H_i \bigcap T_i$).

In fact, this is where the inaccuracy of the *S-P* method comes from, since it uses the heuristic

$$A_{SP}(conn) = A_{ACC}(conn_i^H) \cdot A_{ACC}(conn_i^T), \qquad (1)$$

instead of the accurate

$$A_{ACC}(conn) = A_{ACC}(conn_i^H) \cdot A_{ACC}(conn_i^T | conn_i^H) \qquad (2)$$

$$( = A_{ACC}(conn_i^T) \cdot A_{ACC}(conn_i^H | conn_i^T)).$$

In the sections where we examine state dependencies we will show figures (Fig. 1,2,3) illustrating the availability in two-dimensional probability square. The two dimensions of these squares correspond to the state and availability of $conn_i^H$ and $conn_i^T$. Then, the availability of the whole connection can be evaluated merely by accumulating the rectangular areas where both of $conn_i^H$ and $conn_i^T$ are available. The only difference between evaluating $A_{ACC}$ and $A_{SP}$ is that in case of dealing with $A_{ACC}$ in the dimension of $conn_i^T$ we will use different conditional availability metrics depending on the state of the *not independent* (i.e., overlapping) components.

Our last assumption is that state of the connection is a monotonous function of the state of its components. In other words, improving the state of any component from failure to operational cannot deteriorate the state of the connection, and taking any component into down state cannot result in repaired connection.

## III. Single link overlap

First, we examine a simple connection in which there is a single link overlap when split into two at the $i^{\text{th}}$ node. The common link is $e_{h^*} = e_{t^*}$.

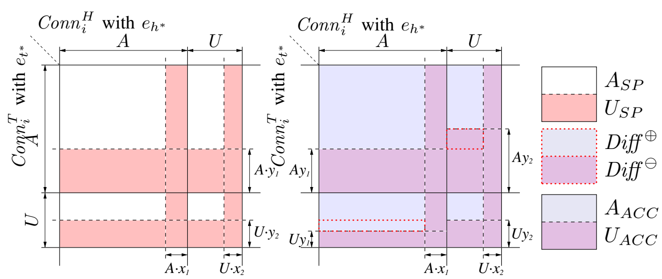



Fig. 1. Visual representation of the estimated and the exact availability and their difference

Figure 1 compares the availability got by the serial-parallel method ($A_{SP}$) to the exact availability ($A_{ACC}$). For the sake of simplicity we use the following notation:

- $A$ = $P(S(e_{h^*}) = 1) = P(S(e_{t^*}) = 1)$ – the availability metric of the investigated link.
- $U$ = $P(S(e_{h^*}) = 0) = P(S(e_{t^*}) = 0)$ – the unavailability metric of the investigated link.
- $x_1$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 1)$ – is the ratio of "bad" states of $conn_i^H$ (when the connection is not available) in case of link $e_{h^*}$ is up.
- $x_2$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 0)$ is the probability that $conn_i^H$ is down in case of link $e_{h^*}$ is down.
- $y_1$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 1)$ is the probability that $conn_i^T$ is down in case of link $e_{t^*}$ is up.
- $y_2$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 0)$ is the probability that $conn_i^T$ is down in case of link $e_{t^*}$ is down.

We introduce two derived metrics which help us to express the difference between $A_{SP}$ and $A_{ACC}$:

$$d_x = x_2 - x_1,$$
$$d_y = y_2 - y_1,$$

indicating how much does $conn_i^H$ depend on $e_{h^*}$ and $conn_i^T$ on $e_{t^*}$.

Note that $A+U = 1$ as they are complementary probabilities and we assume that $x_1 \leq x_2$ and $y_1 \leq y_2$ (implying $d_x, d_y \geq 0$) due to the monotonity assumption – as the probability of connection unavailability if a link is down is always higher or equal compared to the case when that link is up.

Figure 1 also shows how the availability probability metrics ($A_{ACC}$, $A_{SP}$) of the connection can be calculated in a geometric way:

$$A_{ACC} = A(1-x_1)(1-y_1) + U(1-x_2)(1-y_2) \quad (3)$$

$$A_{SP} = \big(A(1-x_1) + U(1-x_2)\big)\cdot \big(A(1-y_1) + U(1-y_2)\big) \quad (4)$$

and how much is the difference between them:

$$\mathit{Diff} = A_{ACC} - A_{SP} = \mathit{Diff}^{\oplus} - \mathit{Diff}^{\ominus} \quad (5)$$

This difference is of high importance in our analysis, so we extract it:

$$\mathit{Diff}^{\oplus} = A(y_2 - y_1)\cdot U(1-x_1) \quad (6)$$
$$\mathit{Diff}^{\ominus} = U(y_2 - y_1)\cdot A(1-x_2) \quad (7)$$

resulting in

$$\begin{aligned} \mathit{Diff} &= \mathit{Diff}^{\oplus} - \mathit{Diff}^{\ominus} \\ &= AU(y_2 - y_1)(x_2 - x_1) \\ &= AU d_x d_y \quad (8) \end{aligned}$$

Note that *Diff* is non-negative as all of its coefficients are also non-negative.

This way, for single link overlap we have proven that the estimated availability is “conservative”, meaning that it is never greater than the exact availability.

The next step is to find the maximal amount of the inaccuracy. As both the real and the approximated connection availability metrics (Equations (3) and (4)) are $\approx 1$ values, the quotient of the two metric is also approximately 1 or near to 1, which does not express significantly the inaccuracy of the *S-P* method. Hence, we define the *divergence* of the *S-P* method as the quotient of the approximated and real *unavailabilities*:

$$DIV_U = \frac{U_{ACC}}{U_{SP}} \quad (9)$$

The difference of unavailabilities is the inverted difference of availabilities. By transforming (5) we get:

$$U_{ACC} = U_{SP} - \mathit{Diff}. \quad (10)$$

Substituting (10) into (9) results in

$$\begin{aligned} DIV_U &= \frac{U_{SP} - \mathit{Diff}}{U_{SP}} \\ &= 1 - \frac{\mathit{Diff}}{U_{SP}}. \quad (11) \end{aligned}$$

We already know that $DIV_U \leq 1$. Now we define a lower bound for $DIV_U$. We state that

$$DIV_U \geq 1 - \frac{d_x d_y}{d_x + d_y} \quad (12)$$

and in the followings we will prove it.

Equation (12) is equivalent with

$$\frac{\mathit{Diff}}{U_{SP}} \leq \frac{d_x d_y}{d_x + d_y}, \quad (13)$$

which can be transcribed into

$$\begin{aligned} \mathit{Diff}(d_x + d_y) &\leq U_{SP} d_x d_y \quad (14) \\ AU d_x d_y (d_x + d_y) &\leq U_{SP} d_x d_y \\ AU(d_x + d_y) &\leq U_{SP} \quad (15) \end{aligned}$$

and relation (15) is confirmed by geometric representation of $U_{SP}$. To prove it also mathematically, we introduce a lower bound of $U_{SP}$.

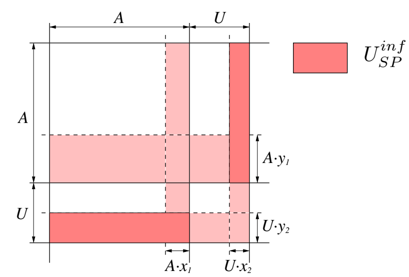



Fig. 2. Lower bound of $U_{SP}$

Figure 2 shows an area $U_{SP}^{inf}$

$$U_{SP}^{inf} = AUy_2 + AUx_2 = AU(x_2 + y_2), \quad (16)$$

which is a part of $U_{SP}$, thus

$$U_{SP}^{inf} < U_{SP}. \quad (17)$$

Applying the inequalities $d_x \leq x_2$, $d_y \leq y_2$, we can write:

$$AU(d_x + d_y) \leq AU(x_2 + y_2). \quad (18)$$

Putting together (18), (16) and (17), respectively, we get

$$AU(d_x + d_y) \leq AU(x_2 + y_2) = U_{SP}^{inf} < U_{SP}, \quad (19)$$

where we find exactly (15) at the left and the right end. Therefore, (12) is also always true.

Putting together the lower and the upper bound estimations we get

$$1 - \frac{d_x d_y}{d_x + d_y} \leq DIV_U \leq 1. \quad (20)$$

We get rougher but simpler lower bound estimation by exploiting that – analogously to the aggregate resistance of parallel switched devices – $\frac{d_x d_y}{d_x + d_y} \leq \min(d_x, d_y)$ and $\min(d_x, d_y) \leq \min(x_2, y_2)$

$$1 - \min(x_2, y_2) \leq DIV_U \leq 1. \quad (21)$$

Recalling the meaning of $x_2$ and $y_2$, (21) formalizes that the exact unavailability of the connection is not only lower but also close to the approximated value: the deviation between the exact and the approximated connection unavailability is never worse than the unavailability of the more viable connection-part (head or tail of the connection) in case the overlapping component is down. And as the connection is protected, even in cases like that, its availability is still high; the unavailability is in order of magnitude of an unprotected link. Thus:

$$\lim_{\forall e \in E : P(S(e)=1) \to 1} DIV_U = 1, \tag{22}$$

expressing that as the components are getting more and more available, the approximation is getting the more accurate.

## IV. Multiple link overlap

In case of multiple overlapping links we want to get answer to the following questions:

- Is $DIV_U \leq 1$?
- Can we find a lower bound for $DIV_U$ similar to the single link overlap scenario?
- we have proven that $DIV_U \to 1$ if all the links are protected and the link availability $\to 1$ ($\lim_{\forall e : P(S(e)=1) \to 1} DIV_U = 1$). Where does $DIV_U$ converge in case of multiple overlaps?

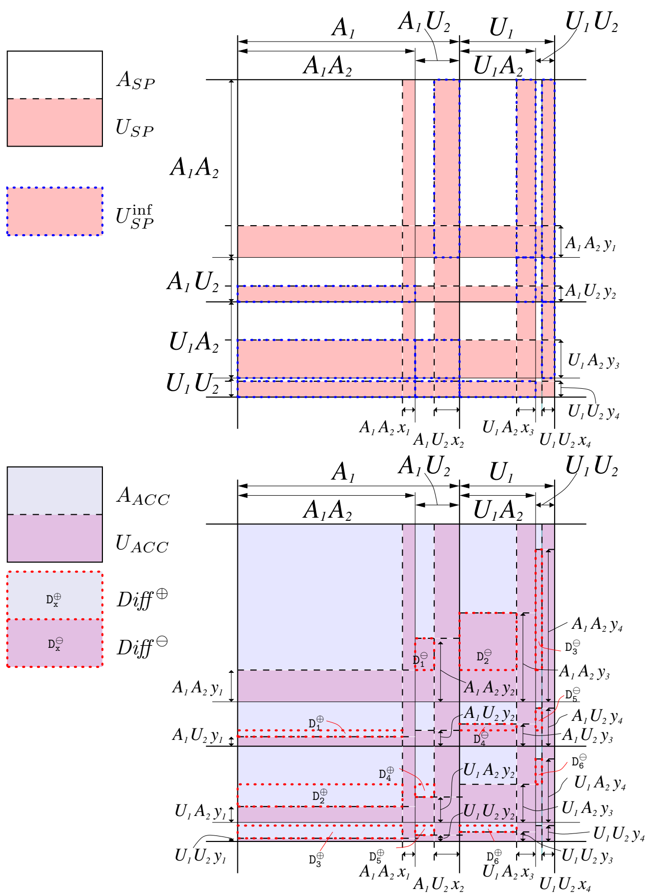



Fig. 3. Visual representation of the inaccuracy of the *S-P* method in case of double link overlap

We recall Fig. 1 and extend it first to double overlap scenario (with overlapping edges $e_{h^*} = e_{t^*}$ and $e_{h^\circ} = e_{t^\circ}$). In Fig. 3 we see a general case with the following notation:

$A_1$ = $P(S(e_{h^*}) = 1) = P(S(e_{t^*}) = 1)$ – the availability metric of the first overlapping link.
$U_1$ = $P(S(e_{h^*}) = 0) = P(S(e_{t^*}) = 0)$ – the unavailability metric of the first overlapping link.
$A_2$ = $P(S(e_{h^\circ}) = 1) = P(S(e_{t^\circ}) = 1)$ – the availability metric of the second overlapping link.
$U_2$ = $P(S(e_{h^\circ}) = 0) = P(S(e_{t^\circ}) = 0)$ – the unavailability metric of the second overlapping link.
$x_1$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 1, S(e_{h^\circ}) = 1)$.
$x_2$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 1, S(e_{h^\circ}) = 0)$.
$x_3$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 0, S(e_{h^\circ}) = 1)$.
$x_4$ = $P(S(conn_i^H) = 0 | S(e_{h^*}) = 0, S(e_{h^\circ}) = 0)$.
$y_1$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 1, S(e_{t^\circ}) = 1)$.
$y_2$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 1, S(e_{t^\circ}) = 0)$.
$y_3$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 0, S(e_{t^\circ}) = 1)$.
$y_4$ = $P(S(conn_i^T) = 0 | S(e_{t^*}) = 0, S(e_{t^\circ}) = 0)$.

Additionally, for each $i \neq j$ pair, we will denote

$$d_{xij} = x_i - x_j, \text{ and}$$
$$d_{yij} = y_i - y_j.$$

Due to the monotonity assumption, the following inequities are true regarding the $x$ values:

$$x_1 \leq x_2 \leq x_4; \quad x_1 \leq x_3 \leq x_4 \text{ and}$$
$$y_1 \leq y_2 \leq y_4; \quad y_1 \leq y_3 \leq y_4.$$

That inequities restrict the range of some $d$ values:

$$i = 4 \vee j = 1 \Rightarrow d_{xij} \geq 0 \wedge d_{yij} \geq 0.$$

Nevertheless, we must emphasize that we cannot state anything concerning the signedness of $d_{x32}$ and $d_{y32}$! And this absence makes the examination difficult.

First, we want to answer the question whether $DIV_U \leq 1$. To do this, we have to collect the pieces of $Diff^{\oplus}$ and $Diff^{\ominus}$. They are:

$$\begin{aligned} Diff^{\oplus} &= \mathtt{D}_1^{\oplus} + \mathtt{D}_2^{\oplus} + \mathtt{D}_3^{\oplus} + \mathtt{D}_4^{\oplus} + \mathtt{D}_5^{\oplus} + \mathtt{D}_6^{\oplus} \\ &= A_1U_2d_{y21} \times A_1A_2(1 - x_1) + \\ &+ U_1A_2d_{y31} \times A_1A_2(1 - x_1) + \\ &+ U_1U_2d_{y41} \times A_1A_2(1 - x_1) + \\ &+ U_1A_2d_{y32} \times A_1U_2(1 - x_2) + \\ &+ U_1U_2d_{y42} \times A_1U_2(1 - x_2) + \\ &+ U_1U_2d_{y43} \times U_1A_2(1 - x_3) \end{aligned} \tag{23}$$

$$\begin{aligned} Diff^{\ominus} &= \mathtt{D}_1^{\ominus} + \mathtt{D}_2^{\ominus} + \mathtt{D}_3^{\ominus} + \mathtt{D}_4^{\ominus} + \mathtt{D}_5^{\ominus} + \mathtt{D}_6^{\ominus} \\ &= A_1A_2d_{y21} \times A_1U_2(1 - x_2) + \\ &+ A_1A_2d_{y31} \times U_1A_2(1 - x_3) + \\ &+ A_1A_2d_{y41} \times U_1U_2(1 - x_4) + \\ &+ A_1U_2d_{y32} \times U_1A_2(1 - x_3) + \\ &+ A_1U_2d_{y42} \times U_1U_2(1 - x_4) + \\ &+ U_1A_2d_{y43} \times U_1U_2(1 - x_4) \end{aligned} \tag{24}$$

Putting (23) and (24) together and replacing $(1-x_i)-(1-x_j)$ by $d_{xji}$ we get:

$$\begin{aligned} \textit{Diff} &= \textit{Diff}^{\oplus} - \textit{Diff}^{\ominus} = \\ &= A_1^2A_2U_2d_{y21}d_{x21} + A_1U_1A_2^2d_{y31}d_{x31} + \\ &+ \quad A_1U_1A_2U_2(d_{y41}d_{x41} + d_{y32}d_{x32}) + \\ &+ \quad A_1U_1U_2^2d_{y42}d_{x42} + U_1^2A_2U_2d_{y43}d_{x43} \end{aligned} \tag{25}$$

Recalling (9) stating that

$$DIV_U = \frac{U_{ACC}}{U_{SP}} = 1 - \frac{\textit{Diff}}{U_{SP}},$$

we can prove that $DIV_U \leq 1$ by proving that $0 \leq \textit{Diff}$.

The only member of (25) which is not evidently non-negative is $A_1U_1A_2U_2(d_{y41}d_{x41} + d_{y32}d_{x32})$, since both $d_{y32}$ and $d_{x32}$ may be negative. Fortunately, for the *absolute values* of the differences, the relations $|d_{y41}| \geq |d_{y32}|$ and $|d_{x41}| \geq |d_{x32}|$ are true. That way

$$d_{y41}d_{x41} + d_{y32}d_{x32} \quad \geq \quad d_{y41}d_{x41} - |d_{y32}d_{x32}| \quad \geq \quad 0.$$

This means that $DIV_U \leq 1$ even in case of multiple link overlaps. In other words, the *S-P* method does not overestimate the connection availability.

The next question is how much is the *positive* divergence of $U_{SP}$. We can transform the task of defining a lower bound of $DIV_U$ into the task of defining an upper bound of $\frac{\textit{Diff}}{U_{SP}}$. We approximate $U_{SP}$ with $U_{SP}^{\inf}$ which is not greater than $U_{SP}$.

$$\frac{\textit{Diff}}{U_{SP}} \leq \frac{\textit{Diff}}{U_{SP}^{\inf}} \tag{26}$$

Figure 3 helps us to extract $U_{SP}^{\inf}$:

$$\begin{aligned} U_{SP}^{\inf} =& A_1^2A_2U_2(x_2+y_2) + A_1U_1A_2^2(x_3+y_3) + \\ &+ A_1U_1A_2U_2(\max(x_3,y_2) + \max(y_3,x_2) + x_4 + y_4) + \\ &+ A_1U_1U_2^2(x_4+y_4) + U_1^2A_2U_2(x_4+y_4) \end{aligned} \tag{27}$$

Note that, opposed to the annotated area the figure, we use $\max(x_3, y_2)$ instead of $x_3$ (and $\max(y_3, x_2)$ instead of $y_3$) to maximize $U_{SP}^{\inf}$ when set against the *Diff* metric.

To be able to compare with *Diff* we introduce $U_{SP}^{\inf *} \leq U_{SP}^{\inf}$ as

$$\begin{aligned} U_{SP}^{\inf *} =& A_1^2A_2U_2(d_{x21}+d_{y21}) + A_1U_1A_2^2(d_{x31}+d_{y31}) + \\ &+ A_1U_1A_2U_2( \\ &\quad \max(x_3,y_2) + \max(y_3,x_2) + d_{x41} + d_{y41}) + \\ &+ A_1U_1U_2^2(d_{x42}+d_{y42}) + U_1^2A_2U_2(d_{x43}+d_{y43}) \end{aligned} \tag{28}$$

Now, first we will give an evident lower bound estimation:

$$\frac{1}{2} \quad \leqq \quad DIV_U \tag{29}$$

which can be transformed into

$$\frac{\textit{Diff}}{U_{SP}} \quad \leqq \quad \frac{1}{2} \tag{30}$$

$$2 \cdot \textit{Diff} \quad \leqq \quad U_{SP}. \tag{31}$$

In (31) we can find for each member of *Diff* the corresponding member in $U_{SP}^{\inf *}$ so that (e.g., for the first member):

$$\begin{aligned} 2 \cdot A_1^2A_2U_2d_{y21}d_{x21} &\leqq A_1^2A_2U_2(d_{x21}+d_{y21}) \\ d_{y21}d_{x21} + d_{y21}d_{x21} &\leqq d_{x21} + d_{y21} \\ 0 &\leq d_{x21}(1-d_{y21}) + d_{y21}(1-d_{x21}), \end{aligned} \tag{32}$$

where in the last line all the members on the right side are non-negative, for that reason (32) is always true.

Note that in (28) we still employ $\max(x_3, y_2)$ and $\max(y_3, x_2)$. The relation of the corresponding *Diff* and $U_{SP}$ parts to be proven is:

$$\begin{aligned} 2 \cdot A_1U_1A_2U_2d_{y32}d_{x32} &\leqq A_1U_1A_2U_2( \\ &\quad \max(x_3,y_2) + \max(y_3,x_2)) \\ 2 \cdot d_{y32}d_{x32} &\leqq \max(x_3,y_2) + \max(y_3,x_2) \end{aligned} \tag{33}$$

This relation is evidently true if one of $d_{y32}$ and $d_{x32}$ is negative, since the left hand side of the relation will be less than zero. If $d_{y32}, d_{x32} \geq 0$, we substitute $d_{y32} + d_{x32} \leq x_3 + y_3 \leq \max(x_3, y_2) + \max(y_3, x_2)$ on the right side and apply (32) as regular, otherwise, if $d_{y32}, d_{x32} \leq 0$, meaning that $y_3 \leq y_2$ and $x_3 \leq x_2$, we can use the $|d_{y32}| + |d_{x32}| \leq y_2 + x_2 \leq \max(x_3, y_2) + \max(y_3, x_2)$ substitution on the right side of the relation. Finally we get

$$2 \cdot d_{y32}d_{x32} \leq |d_{y32}| + |d_{x32}|$$

which relation is valid. That way we have proven that (31) is valid implying that the original assumption in (29) was right.

We can achieve, however, even a much closer lower bound estimation if we define a limit onto the $d$ values:

$$\forall i, j : |d_{xij}|, |d_{yij}| < \varepsilon. \tag{34}$$

If (34) is true, the lower bound is

$$DIV_U \geq 1 - \frac{1}{2}\varepsilon \tag{35}$$

because after similar extraction as before,

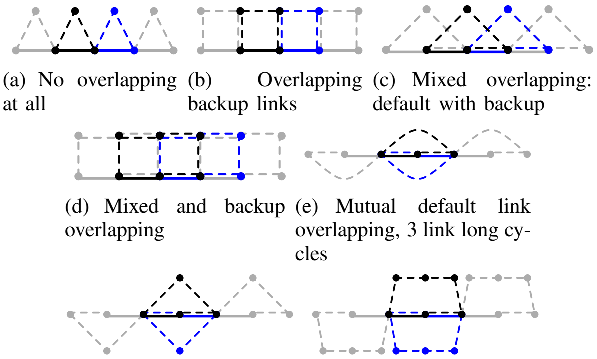
$$\begin{aligned} \frac{\textit{Diff}}{U_{SP}} \quad &\leqq \quad \frac{\varepsilon}{2} \\ 2 \cdot \textit{Diff} \quad &\leqq \quad \varepsilon \cdot U_{SP}, \end{aligned}$$

instead of (32) we will get now ordered pairs like this:

$$\begin{aligned} 2 \cdot A_1^2A_2U_2d_{y21}d_{x21} &\leqq \varepsilon A_1^2A_2U_2(d_{x21}+d_{y21}) \\ d_{y21}d_{x21} + d_{y21}d_{x21} &\leqq \varepsilon(d_{x21} + d_{y21}) \\ 0 &\leq d_{x21}(\varepsilon-d_{y21}) + d_{y21}(\varepsilon-d_{x21}), \end{aligned} \tag{36}$$

where the members on the right side are still non-negative.

However, there are cases when the conditional unavailabilities do not converge to 0. In those cases neither $DIV_U$ will converge to 0. To analyze the convergence of $DIV_U$ we examined some scenarios.



(a) No overlapping at all (b) Overlapping backup links (c) Mixed overlapping: default with backup

(d) Mixed and backup overlapping (e) Mutual default link overlapping, 3 link long cycles

(f) Mutual default link overlapping, 4 link long cycles (g) Mutual default link overlapping, 6 link long cycles

Fig. 4. Different link protection overlap scenarios

## V. Illustrative Examples

Figure 4 depicts seven basic scenarios. In each example the connection is 10 hop long and within a connection the links – or in the last three scenarios the link pairs – of the default path are protected in the same manner. The figures show the neighboring default links with their protection (dashed lines) and the protection overlaps. As we reflect on them in the calculations, we introduce the following additional notation for the components:

- $w_i^r$ is the $i^{\text{th}}$ link on the default path,
- $B_i^r$ is the set protection path of $w_i^r$, consisting of
- $p_{i,j}^r$ links.
- $\varepsilon_u$ is the order of magnitude of the link unavailabilities: $\forall e \in E : c_1 \cdot \varepsilon_u < U(e) < c_2 \cdot \varepsilon_u$. Note that unprotected link chains inherit this order of magnitude – e.g., $U(B_i^r) = \prod_j U(p_{i,j}^r) = O(\varepsilon_u)$.

The simulations carried out on the examples to define the value of $DIV_U$ assumed that each link in the network has the same availability metric ($c_1 = c_2$). The results are shown in Fig. 5.

Scenario (a) does not have any overlapping links at all, scenario (b) contains only backup link overlapping. In these basic examples the conditional unavailabilities ($x$ and $y$ values) converge to 0, this way $DIV_U \to 1$.

In scenarios (c) and (d) the protection of $w_i^r$ leads over $w_{i+1}^r$: $p_{i,1}^r = w_{i+1}^r = e_{h^*}$; in (d) even $p_{i,3}^r = p_{i+1,4}^r = e_{h^\circ}$. In this case if both commonly used links are down, the connection, more closely the connection part $conn_i^T$, becomes unavailable. This is expressed by the coefficient $y_4 = 1$. $y_4 = 1$ implies that $d_{y41}, d_{y42}, d_{y43} \to 1$, which means that there cannot be found any $\varepsilon < 1$ for $DIV_U$, hereby the lower bound for the deviation is $\frac{1}{2} \leq DIV_U$.

Still, $DIV_U$ converges to 1. And the reason for this behaviour is the following: all the remaining conditional probabilities of $x_i$ and $y_j$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$(!) will converge to 0, since they refer to states probabilities of $conn^H$ or $conn^T$, when either the working path or the protection path (or both) are available. All these cases imply a conditional unavailability of order $\varepsilon_u$: $x_i, y_j \leq c_3 \cdot \varepsilon_u$ for $i \in \{1, 2, 3, 4\}, j \in \{1, 2, 3\}$. This upper boundary ($c_3 \cdot \varepsilon_u$) is inherited also by the remaining $d$ values – all except for the previously mentioned $d_{y41}, d_{y42}, d_{y43}$.

Without the loss of generality, but for the sake of simple calculations, we assume that $c_2 < c_3$. This way, substituting $c_3 \cdot \varepsilon$ values into Eq. (25) we get that $\mathit{Diff} < 6 \cdot c_3^3 \cdot \varepsilon^3$. Whereas – after applying the lower bound of $U_{SP} > U_1 U_2 y_4$ taken from Fig. 3 –, $U_{SP} > c_1^2 \cdot \varepsilon^2$. Thus the quotient of $\mathit{Diff}$ and $U_{SP}$ is

$$\frac{\mathit{Diff}}{U_{SP}} < \frac{6 \cdot c_3^3 \cdot \varepsilon^3}{c_1^2 \cdot \varepsilon^2} < 6 \cdot c_3^3 \cdot c_1^{-2} \cdot \varepsilon,$$

which also converges to 0. The simulation results in Fig. 5 confirm this reasoning.

Scenarios (e), (f) and (g) are more complicated than the former ones as the backup paths of $w_i^r$ and $w_{i+1}^r$ mutually overlap the default link of each other ($w_i^r = e_{h^*}$ and $w_{i+1}^r = e_{h^\circ}$). The question is where do their $DIV_U$ values converge. On the one hand, we can observe that if both overlapped links are down, the connection, precisiously said, the *half-connections* $conn_i^H$ and $conn_i^T$ become unavailable ($x_4 = y_4 = 1$). On the other hand, however, if the default path is available we do not care about the state of the backup path (in case of $S(e_{h^*}) = 1$ we use $x_d = x_1 = x_2 = U_{ACC}(conn_{i-1}^H)$ and in case of $S(e_{t^\circ}) = 1$ we use $y_d = y_1 = y_3 = U_{ACC}(conn_{i+1}^T)$). These values were easy to derive.

The remaining probabilities, i.e., when the backup paths are only available, are somewhat more complicated to evaluate. Now we introduce the synonyms $x_b = x_3$ and $y_b = y_2$ to denote these *backup* unavailabilities.

Both $x_d$ and $y_d$ are $O(\varepsilon^2)$, since the connection unavailability depends in these cases only on remaining connection segments $conn_{i-1}^H$ and $conn_{i+1}^T$, and these segments are single protected resulting in unavailability of $O(\varepsilon_u^2)$. For $x_b$ and $y_b$ the connection segments contain an unprotected link chain – practically, the backup path is operational and it is not protected – thus they are $O(\varepsilon)$.

Knowing that $U_{SP}$ is $O(\varepsilon^2)$, we want to define rather a more expressive metric that converges to a constant non-zero value, i.e., is of order $\frac{U_{SP}}{\varepsilon^2}$:

$$\begin{aligned}
\lim_{\varepsilon \to 0} \frac{U_{SP}}{U_1 U_2} &= \lim_{\varepsilon \to 0} \frac{A_1 U_2 y_b + U_1 U_2 + U_1 A_2 x_b + U_1 U_2}{U_1 U_2} + \\
&\quad + \frac{(A_1 U_2 y_b + U_1 U_2) \cdot (U_1 A_2 x_b + U_1 U_2)}{U_1 U_2} \\
&= 2 + \frac{U_2 y_b + U_1 x_b}{U_1 U_2} \\
&= 2 + \frac{y_b}{U_1} + \frac{x_b}{U_2}. \qquad (37)
\end{aligned}$$

Regarding the $\mathit{Diff}$ metric our starting point is Eq. (25). We already showed that $d_{x4i}$ and $d_{y4i}$ values converge to 1, whereas the other $d$ values are $O(\varepsilon^n)$ where $n$ is 1 or higher. This way there will remain only one member of the sum $\mathit{Diff}$ which is $O(\varepsilon^2)$ making the limit value calculation easy for the

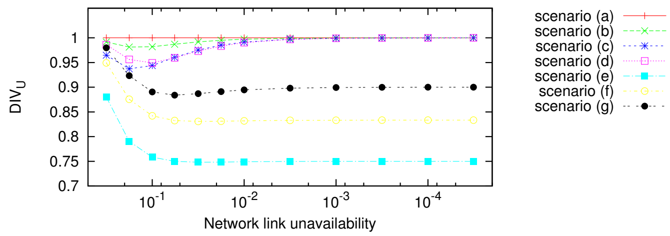



Fig. 5. Inaccuracy of the Serial-Parallel model in case of multiple link overlaps

following expression:

$$\lim_{\varepsilon\to 0}\frac{Diff}{U_1U_2} = \lim_{\varepsilon\to 0}\frac{U_1U_2d_{x41}d_{y41}}{U_1U_2}+\varepsilon\cdot rest = \lim_{\varepsilon\to 0} d_{x41}d_{y41} = 1. \quad (38)$$

Putting (37) and (38) together, fiinally we get that

$$\lim_{\varepsilon\to 0} DIV_U = 1-\frac{1}{2+\frac{y_b}{U_1}+\frac{x_b}{U_2}}. \quad (39)$$

The lines corresponding to the last three scenarios in Fig. 5 confirm these calculations: revoking that $U_1 \approx U_2 \approx \varepsilon_u$, in scenario (e) the cycles are 3 link long, this way $y_b \approx U_1$ and $x_b \approx U_2$. $DIV_U$ will converge to $1-1/(2+1+1) = 1-1/4 = 3/4$. In scenario (f) the cycles are 4 link long, this way $y_b \approx 2U_1$ and $x_b \approx 2U_2$. $DIV_U$ will converge to $1-1/(2+2+2) = 1-1/6 = 5/6$. Finally, in scenario (g) the cycles are 6 link long, this way $y_b \approx 4U_1$ and $x_b \approx 4U_2$. $DIV_U$ will converge to $1-1/(2+4+4) = 1-1/10 = 0.9$.

## VI. Conclusion and Application

In this paper we have examined the accuracy of the Serial-Parallel aggregated availability calculation method. In case the failure states of the network components in the default and backup paths are independent, the *S-P* method is accurate. However, overlapping components, e.g., network links member of multiple link/segment protection paths, the calculation "cheats". Using geometric representation of probability space we have examined single and multiple link overlaps. We have proven that

- the accurate unavailability of a connection ($U_{ACC}$) is always lower than the unavailability approximated by the serial-parallel method ($U_{SP}$);
- there can be defined lower bounds of the divergence expressed by $DIV_U = U_{ACC}/U_{SP}$:

$$DIV_U \geq \begin{cases} 1-min(d_x,d_y), & \text{for single link overlap} \\ 1-\frac{1}{2}\varepsilon, & \text{for dual (or more) link} \\ & \text{overlaps and } \forall d_{xij},d_{yij}:\varepsilon \geq d_{xij},d_{yij} \end{cases}$$

- By increasing the link availability metric in the network, i.e., $\forall e: P(S(e)=0) \leq \varepsilon$, the divergence of the calculated and the accurate connection unavailability converges to a defined value:

$$\lim_{\varepsilon\to 0} DIV_U = \begin{cases} 1, & \text{if there are no mutual working link overlaps} \\ 1-\frac{1}{2+\frac{y_b}{U_1}+\frac{x_b}{U_2}}, & \text{if the protections of neighboring working links mutually overlap each other} \end{cases}$$

The presented theoretical result can be applied in several situations. For example, the different protection alternatives of the same connection can be partially ranked by their availability, as long as difference between the results of the *S-P* approximation are higher than the worst case inaccuracy of the *S-P* method. We achieved further applicability in developing most available path searching algorithm for link-protected (e.g., *p*-cycle-protected) connections. With the help of the *S-P* approximation and knowing the upper bound of its divergence, we can find a metric which is higher than the real availability metric. In our application we use that metric as hint within an $A^*$-algorithm.

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